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T W O
MATHEMATICAL ESSAYS:
THE FIRST
ON ULTIMATE RATIOS,
THE SECOND
ON THE POWER OF THE WEDGE.

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The substance of the following Essays was part of a course of lectures delivered in St. John's College, Cambridge: but having been imperfectly taken down by some of the hearers, the author thought proper to reduce it into writing himself, and has been prevailed upon to make it public.

ESSAY I.

On ULTIMATE RATIOS.

I. **A** Great part of the labour of geometricians has been employed in comparing the areas of curvilinear figures with rectilinear figures, and the areas of curvilinear figures with each other; not merely on account of the importance of the subject itself, but because almost all questions in philosophy can be reduced to a comparison of such areas. For instance; there are no two forces in mechanics however related to each other, but geometricians can feign two figures so circumstanced that their areas shall have the same proportion to each other as the proposed forces; therefore any theorems serving to discover the unknown proportion of those geometrical figures from the assumed circumstances, will discover likewise the proportion of the quantities analogous to them. An example of this

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we have in the doctrine of falling bodies, in which the proportion of the right lines described in their descent when the times of their fall are different, is found by comparing the areas of two right angled triangles. For if the perpendicular heights of those triangles be assumed proportional to the respective times of descent and their bases to the velocities acquired in the fall, it is demonstrable that the right line described in their descent will be proportional to the areas of those triangles; and thus the right line described by heavy bodies in their fall, is said to be *represented* by the area of a right angled triangle.*

2. We find Newton frequently prefacing his problems with this expression, *Concessis figurarum curvilinearum quadraturis, requiritur, &c.* that is by feigning some geometrical figure whose area he will show to be analogous to the quantity sought, the problem is reduced to a geometrical one. By this means we are assured that the conditions of the problem involve no inconsistency but that it is possible; and if the art of comparing curvilinear figures be sufficiently

* Keil's Physics Lect. XI. Theorem XVII. Cotes *De descensu gravium*. Mac Laurin's Account, &c. B. II. Chap. 5.

ently understood, the solution itself may be drawn from hence, and perhaps many other circumstances may be inferred, which otherwise would not easily appear.

3. The primary method of comparing the magnitude of rectilinear figures, is by laying them upon one another, it being an axiom that such as agree in all their parts are equal.* Thus all the right lined figures which in the first book of the *Elements* are proved to be equal, are always shown to be the sum or difference of such as would respectively cover each other. This is the direct and best way of reasoning, in which the agreement of the extreme ideas is shown by their continual agreement with intermediate ideas of the same kind: but this method cannot be applied in comparing curvilinear figures with rectilinear ones, because no part of a curve line, can ever coincide with a straight line.

4. To extend geometry to the comparing curvilinear with rectilinear figures the ancients made use of the method of *Exhaustions*; † an example of which we have in the second proposition of the twelfth book

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* Axiom 8 and prop. 4. El. 1. † See *Archimedes de dimensione circuli*, &c.

of Euclid. I shall only observe upon it, that the argument made use of is what is called *reductio ad absurdum*, which though strictly logical is always tedious, inasmuch as every proposition must be divided into two cases, in one of which you are to show that the former of the quantities to be compared together is not greater than the latter, and in the other that it is not less.

5. To contract this method of reasoning, *Cavalierius* proposed what is called the method of *Indivisibles*,* in which he was followed by Dr. *Wallis* and others of the last century. In this method every line is supposed to consist of a number of other lines of the *smallest possible* length; every curve was considered as a polygon each of whose sides is one of those *indivisible* lines; with other like suppositions equally absurd and ungeometrical. These principles soon led their followers into perplexity and oft-times into error, nor was it easy to fix bounds to those liberties when once introduced.

6. To avoid both the tediousness of the antients and the inaccuracy of the moderns, Sir Isaac Newton introduced what he called
the

* See *Geometria indivisibilibus promota*. Ed. 1635.

the method of *prime and ultimate ratios*, the foundation of which is contained in the first Lemma of the first book of the *Principia*. Many difficulties have been started and much controversy raised concerning the proof of it; * all which would have been avoided had either the author or his readers observed that he is in reality laying down the *definition* of a term, (namely, being *Ultimately equal*,) and not *proving a proposition*. Taking therefore this first Lemma for a definition it may be explained, (not *proved*) in the manner following.

7. Let there be two quantities, one fixed and the other varying, so related to each other that *first* the varying quantity continually approaches to the fixed quantity. *Secondly*, that the varying quantity does never reach or pass beyond that which is fixed. *Thirdly*, that the varying quantity approaches nearer to the fixed quantity than by any assigned difference, † then is such a fixed quantity called the *limit* of the varying quantity; or in a looser way of speaking, the varying quantity may be said to be *ultimately*

* See several pamphlets written by Mr. *Robins* and Dr. *Jurin* in the years 1736 and 1740, and other papers in the memoirs of literature for those and the intermediate years.

† *Proprius accedunt quam pro data quavis differentia.*

timately equal to the fixed quantity.* These three properties may be otherwise expressed more distinctly thus. *First*, the difference between the varying quantity and the fixed quantity must continually decrease. *Secondly*, This difference must never become either nothing or negative. *Thirdly*, This difference must become less in respect to the fixed quantity than by any assigned ratio; or the difference between the two quantities must become a less part of the fixed quantity than any fractional part that is assigned, how small so ever the fraction expressing such part may be.† Wherever these properties are found, the fixed quantity is called the *limit* of the varying quantity, or the varying quantity is *said* to be *ultimately equal* to the fixed quantity. This last phrase must not be taken in an absolute literal sense, there being no *ultimate state*, no particular magnitude that is the *ultimate magnitude*

* Some writers seem to have dropped the second condition out of their idea of a limit; however Newton plainly supposes it. His words are — *Ultimæ rationes* — *revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant; et quas proprius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, &c.*

Scholium ad Lemma XI. Lib. I. Princip. pag. 38. Ed. 3.

† See the definition of proportion given in *Saunderson's Algebra* art. 15. page 76. Ed. quarto. or p. 89. Ed. octavo.

nitude of such a varying quantity. Under the word quantity in this definition must be included not only numbers, lines, &c. but more especially ratios considered as a peculiar species of quantity; but as the consideration of ratios which have limits is difficult, we shall begin with examples of other quantities.

EXAMPLES IN NUMBERS.

8. Let there be formed a series whose first term is unity, second $\frac{1}{2}$, third $\frac{1}{4}$, fourth $\frac{1}{8}$, and so on, every term being half the preceding term thus ; $1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}$ &c. and let the sum of an indefinite number of terms in this series be considered as continually varying, being continually increased by the accession of a new term; thus the sum of two terms is $1\frac{1}{2}$, of three terms is $1\frac{3}{4}$, of four terms is $1\frac{7}{8}$, &c. I say then that the varying sum of the terms of this series continually approximates to the fixed number 2 as its limit. For the difference between the sum of one, two, three, four, &c. terms and the number 2 will be the

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numbers $1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}$, &c. successively
 in infinitum. Here it is evident that the
 terms of this last series which express the
 successive differences between the growing
 sum of the terms of the former series and
 the number 2, *first* continually decrease;
secondly no term in this series of differences
 can become either nothing or negative;
thirdly we may continue this series of suc-
 cessive differences, 'till we come to a term
 which shall be a less part of the fixed num-
 ber 2, than any fractional part that can be
 assigned, or so that this difference shall be
 less when compared with the number 2, than
 in any proportion assigned. For instance
 we may continue this series of differences,
 'till we come to a term which shall be less
 than a $\frac{1}{1000}$ th part of the number 2, or less
 than the $\frac{1}{10\ 000}$ th part of the number 2,
 or 'till you come to a term which shall be
 less than any other fractional part of the
 number 2 which you please to point out
 and assign. The number 2 then having
 the conditions laid down in the definition
 is to be *called* the limit of the sum of the
 terms

terms of this infinite series $1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}$
 &c. Any infinite series being proposed, if
 a number can be pointed out having the
 conditions before mentioned then such series
 is said to have a limit.

In the usual phrase of mathematicians,
 the number 2 in our example would be
 called the sum of *all* the terms in the pro-
 posed series continued *in infinitum*; but in
 this case the words must not be understood
 in their literal sense, it being absurd to speak
 of the sum of *all* the terms of *an infinite* se-
 ries, that is, of a series which has *no all* to
 its number of terms. When mathemati-
 cians talk of finding the sum of an infinite
 series, they neither mean the sum of any
 particular number of terms, nor yet the
 sum of all the terms, but the *limit* of the sum
 of the terms as before explained.

9. No number less than 2, for instance
 $1\frac{7}{8}$ can be taken for the limit; for in this
 case it will not answer the second condition.
 In this example the sum of four terms of
 the series equals $1\frac{7}{8}$, and the sum of five
 terms exceeds it; therefore the difference
 between this sum and the number $1\frac{7}{8}$
 proposed as a limit is in the former case

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nothing,

nothing, and in the latter negative. No number greater than 2, for instance 3, can be taken for the limit, because the last condition will be wanting. For if the sum of any assigned number of terms be always less than 2, that sum must always want more than unity of the number 3, and consequently cannot approach nearer to 3 than by the assigned difference 1.

10. In like manner the sum of the series

$1 \cdot \frac{1}{3} \cdot \frac{1}{9} \cdot \frac{1}{27}, \&c.$ continued in infinitum

will be $1\frac{1}{2}$; the series of successive differ-

ences being $\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{18} \cdot \frac{1}{54}, \&c.$ in infinitum.

11. To give a similar instance in Geometry; Let AI (in fig. 1.) be an indefinite right line, on which set off the equal parts $AB \cdot BC \cdot CD \cdot DE, \&c.$ on AB make a square, on BC a parallelogram whose breadth Bb is half the side AB , on CD a parallelogram whose breadth Cc is half the breadth of the former parallelogram, on DE another whose breadth is half Cc and so on. It is evident that the right lined figure composed of all these parallelograms though infinite in extent from A towards I ,
is

is yet finite in area, and is equal to twice the square upon AB ; that is, though there is no limit to the extent of such a figure from A towards I , yet there is a limit to its area in our sense of the word limit; nor is it strange that quantities which have no limits in one respect, should yet have limits in another.

12. To make the example in par. 8 general. Let $a : b$ express the common ratio of any series of numbers in continual proportion whose first term is unity; I say then that if a be greater than b , such a series will have a limit, viz. $\frac{a}{a-b}$, or in other words the sum of all the terms of that series continued in infinitum will be $\frac{a}{a-b}$. For the terms of this series will be

$$1 \cdot \frac{b}{a} \cdot \frac{b^2}{a^2} \cdot \frac{b^3}{a^3} \cdot \frac{b^4}{a^4}, \text{ \&c. whence}$$

The sum of 1 term is 1

$$2 \text{ terms is } \dots \frac{a+b}{a}$$

$$3 \text{ terms is } \frac{aa+ab+bb}{aa}$$

$$4 \text{ terms is } \frac{a^3+aab+abb+b^3}{a^3}$$

Let each of these sums be subtracted from the limit $\frac{a}{a-b}$, and we have the successive differences

$$\frac{b}{a-b} \cdot \frac{b}{a-b} \times \frac{b}{a} \cdot \frac{b}{a-b} \times \frac{b^2}{a^2} \cdot \frac{b}{a-b} \times \frac{b^3}{a^3} \cdot$$

&c. or if n be any assigned number of terms, the difference between the sum of that number of terms and $\frac{a}{a-b}$ will be

$$\frac{b}{a-b} \times \left[\frac{b}{a} \right]^{n-1} *$$

Whence we may observe *first* that as the number of terms to be summed up increases, this difference

$$\frac{b}{a-b} \times \left[\frac{b}{a} \right]^{n-1} \text{ continually decreases, because}$$

* The sum of any particular number of terms n may be found thus. Put $x = \frac{b}{a}$ and the series will stand thus,

$$1 + x + x^2 + x^3 + x^4 \dots x^{n-1} \text{ which put } = s$$

whence multiplying by x

$$x + x^2 + x^3 + x^4 \dots x^{n-1} + x^n = +sx$$

subtract the latter step from the former and

$$1 * * * * * - x^n = s - sx = 1 - x \times s$$

$$\text{whence } s = \frac{1-x^n}{1-x} \text{ or restoring } x, s = \frac{a^n - b^n}{a^{n-1} \times a - b}$$

because x is a fraction less than unity, its powers will continually decrease; by increasing n the number x^n may become less than any assigned fraction, that is, when n is in-

$$\text{finite } x^n \text{ vanishes and we have } s = \frac{1}{1-x} \text{ or (restoring } x)$$

$$s = \frac{1}{1-\frac{b}{a}} = \frac{a}{a-b} \text{ as before.}$$

cause $\frac{b}{a}$ being a fraction less than unity, its powers continually decrease. *Secondly*, this difference can never become nothing or negative; the powers of a fraction though they decrease being always real and affirmative. *Thirdly*, this difference may become less in respect of $\frac{a}{a-b}$ than by any assigned proportion. For $\frac{a}{a-b}$ is to $\frac{b}{a-b} \times \left(\frac{b}{a}\right)^{n-1}$ as a^n is to b^n or as $\left(\frac{a}{b}\right)^n$ is to 1; but $\frac{a}{b}$ is greater than unity, therefore the powers of $\frac{a}{b}$ continually increase, and by increasing the index n the number $\left(\frac{a}{b}\right)^n$ may become greater than any assigned number, consequently $\left(\frac{a}{b}\right)^n$ may become greater in respect of 1, or 1 less in respect of $\left(\frac{a}{b}\right)^n$ than by any assigned ratio; therefore the difference $\frac{b}{a-b} \times \left(\frac{b}{a}\right)^{n-1}$ may become a less part of $\frac{a}{a-b}$ than any assigned

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ed fractional part. The quantity $\frac{a}{a-b}$ therefore having these three properties agrees with the definition of a limit of the growing sum of the terms of the series, or we may say (in the language of mathematicians) that it is equal to the sum of all the terms continued in infinitum.

13. What has been thus proved may be shown more concisely by dividing a by $a-b$ in the manner of compound division in arithmetic; * for the quotient will be the very series proposed in the example, and it may be further confirmed by multiplying that series by $a-b$, for the product will be a , all the terms except the first destroying one another; therefore a divided by $a-b$ is equal to that whole series. †

14. We may observe that if every term of the foregoing series be multiplied by any num-

* See Saunderson's Algebra, page 73. Quarto Edition.

† Hence again we may find the sum of n terms in this series. For putting $x = \frac{b}{a}$ as before, the last or n^{th} term will be x^{n-1} , and the remaining terms of the series will be $x^n + x^{n+1} + x^{n+2} + x^{n+3}, \&c. = x^n \times 1 + x + x^2 + x^3 + x^4, \&c. = x^n \times \frac{1}{1-x} = \frac{x^n}{1-x}$; Subtract this from $\frac{1}{1-x}$ or the sum of all the terms, and there will remain the sum of the first n terms $= \frac{1-x^n}{1-x}$ as before.

number c it will become $c + \frac{bc}{a} + \frac{b^2}{a^2} \times c$
 $+ \frac{b^3}{a^3} \times c$, &c. that is, a series of numbers
 in continual proportion whose common
 ratio is that of a to b as before, but whose
 first term is c . It is evident therefore, that in
 this case, to find the sum of all the terms
 we must multiply the former limit by c ,
 that is, it will be $\frac{ac}{a-b}$.

15. *Example 1.* Let the first term be 1,
 and the common ratio 10 to 1, then the
 several terms of the series are $1 \cdot \frac{1}{10} \cdot \frac{1}{100} \cdot$

$\frac{1}{1000}$, &c. or if we express them decimally
 the first term is 1,

2^d 0,1

3rd 0,01

4th 0,001

their sum $\frac{1,111 \text{ \&c.}}{10-1} = \frac{10}{9}$

Example 2. Let the first term be 1, the
 common ratio 9 to 1, and the terms ex-
 pressed decimally are

the

the 1st term 1,

2^d 0,111111

3^d 0,012345

4th 0,001371

5th 0,000152

6th 0,000017

their sum $1,124996 = \frac{9}{8} = 1,125.$

16. Before we proceed to consider ratios that have limits we must learn to distinguish between the terms of a ratio and the ratio itself. The terms of a ratio may vary through all degrees of magnitude, and yet the ratio itself continue invariable. Let AQ (fig. 2.) be an indefinite right line, AI another intersecting it in a given angle at A , and on any point P in AQ draw PM at right angles to AQ meeting AI in M . Let the point P move upon the right line AQ so that AP and PM may increase and decrease. The lines AP and PM in this case may vary through all degrees of magnitude, may become greater or less than any assigned line; that is, in the common language of mathematicians, may become infinitely great or infinitely little, yet the ratio of AP to PM is invariably the same, because the triangles APM are always similar.

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In this instance then, varying the terms does not vary the ratio at all; in others varying the terms in infinitum does likewise vary the ratio in infinitum; and there are instances where although changing the terms does change the ratio, yet the ratio never varies beyond certain limits, even though the terms themselves should increase or decrease in infinitum. The following example of this is borrowed from *Saunderson's algebra*, art. 336.

17. Let x be any varying quantity; make $4xx + 3x = A$ and $2xx + x = B$, then will A and B also be varying quantities as depending upon x ; when x vanishes A and B will both vanish, and when x is infinite they will both be infinite. I say that the ratio of A to B while x decreases in infinitum approximates to the ratio of 3 to 1 as its limit. For *first* A is to B as $4xx + 3x$ is to $2xx + x$ or as $4x + 3$ is to $2x + 1$. Here it is easy to see that as x decreases the quantities $4x$ and $2x$ decrease, and consequently the ratio of $4x + 3$ to $2x + 1$ or A to B approaches to that of 3 to 1. *Secondly*, the ratio of A to B can never reach or exceed that of 3 to 1. For $6xx + 3x$ is to $2xx + x$ as 3 is to 1, but $4xx + 3x$ is

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is

is a less quantity than $6xx + 3x$, therefore $4xx + 3x$ is to $2xx + x$ or A is to B always in a less ratio than that of 3 to 1. *Lastly*, the ratio of A to B will approach nearer to that of 3 to 1 than by any assigned difference. For in the terms of this ratio $4x + 3$ and $2x + 1$, the varying parts $4x$ and $2x$ by diminishing x may become less than any assigned quantity while the other parts 3 and 1 remain the same; therefore the ratio of A to B will approach nearer to the ratio of 3 to 1, than by any assigned difference.

In like manner the ratio of A to B , while x increases in infinitum approximates to the ratio of 2 to 1 as its limit. For since A is

to B as $4x + 3$ is to $2x + 1$ or as $4 + \frac{3}{x}$

is to $2 + \frac{1}{x}$ it is easy to see that as x in-

creases the quantities $\frac{3}{x}$ and $\frac{1}{x}$ decrease and

consequently the ratio of $4 + \frac{3}{x}$ to $2 + \frac{1}{x}$

or A to B approaches to the ratio of 4 to 2 or 2 to 1. Again; the ratio of A to B can never be less (that is nearer to the ratio of equality) than the ratio of 2 to 1. For

$4xx$

$4xx + 2x$ is to $2xx + x$ as 2 is to 1, but $4xx + 3x$ is a greater quantity than $4xx + 2x$; therefore $4xx + 3x$ is to $2xx + x$ or A is to B always in a greater ratio than that of 2 to 1. Lastly; the ratio of A to B will approach nearer to that of 2 to 1, than by any assigned difference. For in the

terms of this ratio $4 + \frac{3}{x}$ and $2 + \frac{1}{x}$,

the variable parts $\frac{3}{x}$ and $\frac{1}{x}$, by increasing x may become less than any assigned fraction while the parts 4 and 2 remain the same; therefore the ratio of A to B will approach nearer to the ratio of 4 to 2 or 2 to 1 than by any assigned difference.

Here then we see that though diminishing x and consequently diminishing the terms A and B increases their ratio, and contrariwise increasing these terms by increasing the quantity x decreases their ratio, yet there is a limit both to the increase and decrease of this ratio, although there is none to the terms themselves that compose it.

18. The ratio of 3 to 1 which limits the ratio of A to B when these terms decrease in infinitum is called the *ultimate ratio* of the evanescent quantities A and B ; and it

is likewise called, but more rarely, the *first ratio* of the nascent quantities A and B . The ratio of 2 to 1 which is their other limit, is called the *ultimate ratio* of the quantities A and B increasing in infinitum.

18. Another example of a similar nature we have as follows. Let x be a varying quantity and d a constant one, then will $x + d$ and x be two varying quantities capable of all degrees of magnitude. I say that the ratio of $x + d$ to x , while x increases will continually decrease, but not beyond a certain limit, which limit is the ratio of equality. On the contrary, if x decrease, the ratio of $x + d$ to x will continually increase more and more in infinitum and never come to a limit. For $x + d$ is to x as

$1 + \frac{d}{x}$ is to 1; now as x increases, the fraction $\frac{d}{x}$ decreases and may become less than

any assigned fraction; but the number 1 which is the other part of the antecedent of this ratio remains the same, so likewise does the consequent of this ratio, therefore

the ratio of $1 + \frac{d}{x}$ to 1 continually approximates to the ratio of equality. *Secondly;*
the

the fraction $\frac{d}{x}$ has always some magnitude, therefore the antecedent $1 + \frac{d}{x}$ is ever greater than the consequent 1; therefore the ratio of $1 + \frac{d}{x}$ to 1 can never reach the ratio of equality, much less pass it so as to become a ratio *minoris inæqualitatis*, or a ratio in which the antecedent is less than the consequent. *Lastly*, the varying fraction $\frac{d}{x}$, as x increases, will become less than any assigned fraction, while the other part of the antecedent and likewise the consequent of this ratio remains the same. Therefore this ratio of $1 + \frac{d}{x}$ to 1 will approach nearer to the ratio of equality than by any ratio that can be assigned, how small soever such ratio may be; therefore the ratio of equality is the limit of the ratio of $1 + \frac{d}{x}$ to 1, and consequently the limit of the ratio of $x + d$ to x which continually decreases while the terms which compose this ratio continually increase in infinitum.

If

If x decreases then $\frac{d}{x}$ will increase and may become greater than any assigned number, how great soever that number may be, while the other part of the antecedent and likewise the consequent remain fixed, each being unity; therefore the ratio of $1 + \frac{d}{x}$ to 1, and consequently the ratio of $x + d$ to x , as x decreases, will become greater than any assigned ratio whatever and so has no limit to its increase. Here we have an instance of a ratio that has a limit to its decrease, but none to its increase.

We may observe that though the terms of this ratio, *viz.* $x + d$ and x , never get nearer to each other, their constant difference being d , yet the ratio of the terms approximates to the ratio of equality; though the terms get no nearer in their difference, yet (if the expression may be allowed) they get nearer in their ratio. The *difference* of the terms and the *ratio* of the terms, are ideas very distinct from each other, and in no wise so connected, but that one may vary while the other is constant. In this instance the ratio varies, but the difference of the terms is constant; on the contrary, the dif-
fer-

ference between the two lines AP and PM in fig. 2. (see par. 16.) continually varies, but the ratio of those lines is invariable.

19. Although the ratio of equality may be strictly called the *limit of the varying ratio* of the quantities $x + d$ and x , yet the terms of this ratio can never be strictly said to be equal, no not *ultimately equal*, since that plainly supposes an *ultimate state* in which they are equal; no nor equal when they *vanish into infinity*, or when they step out of finite existence into infinity. There is no finite quantity next to infinity; no number (for instance) which is the next number to infinity, and therefore no step out of finite into infinite.* Nor can we say in strictness that two infinitely great numbers with a finite difference are equal, it being a proposition plainly absurd and contradictory. There is no such thing in nature as any infinitely great number; and it is contradictory to say of any two numbers, both that they have a difference, and that they are equal. Whoever considers that the idea of infinity is a general or abstract

* Neither is there any step out of a *state of nothingness* into finite existence. There is no fraction so small as to be the *very next* fraction to nothing. No fraction can ever be assigned so small, but another fraction may be assigned that is smaller.

abstract idea, that the idea of number is always particular, that infinity is a *property* of number, a *property* of extension, &c. not any number, not any extension, &c. itself, will easily see that these and such like expressions can have no *literal* meaning.* As for the metaphorical use of them to avoid circumlocution or the introduction of new terms, it may be allowed (when once the literal

* By saying that number or that extension is infinite, we mean only to assert the impossibility of limiting the increase of number or the increase of extension: we mean to assert the absurdity of fixing upon any particular number how great soever, or any particular extension how large soever, as the biggest possible number or the greatest possible extension. There is no such thing existing as a number actually infinite, or an infinite right line. Euclid requires you to allow the possibility of producing a right line as far as ever he is pleased to direct, or as some would say, the possibility of producing it *in infinitum*, but he makes no propositions about infinite right lines — ἐκβαλλόμεναι αἱ δύο αὐταὶ εὐθεῖαι ἐπ' ἄπειρον — in the twelfth axiom which is translated by *Commandine*, *rectæ lineæ illæ in infinitum productæ*, is rendered by *Dr. Simson* — these straight lines being continually produced: So likewise prop. 12. El. 1. Ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον, — translated by *Commandine* — *super data recta linea infinita* — is rendered by *Dr. Simson* — upon a straight line of an unlimited length. The term *infinite* has been so abused that it can hardly be admitted in mathematical writings any longer; and it is high time to drop it, when authors talk of adding, subtracting, &c. *infinites and infinitesimals* as familiarly as if they were common numbers.

It may be proper to observe here, that as number and extension have no limit to their increase, so neither have they any limit to their decrease. It is absurd to fix upon any particular fraction as the least possible number, or to fix on any particular line as the least possible extension — There is *no least possible* — There exists no such thing as a fraction infinitely small, or a line infinitely little; nor did *Euclid* or *Archimedes* or any of the antient geometers ever suppose it.

literal meaning has been explained) on this as well as numberless other occasions, both in science and common life.

20. When the difference between any two quantities decreases so as to become a less fractional part of one of them than any assigned fractional part whatever; or when the difference between the terms of a ratio becomes less in respect of one of them (the greater for instance) than by any assigned ratio (according to the third condition required in par. 7) we shall express this for the future by saying that such difference *vanishes* in respect of that (*greater*) quantity.

21. Now if the difference between two terms vanishes in respect of one of them, it will also vanish in respect of the other. It is true that at any assigned time, when the terms have a particular magnitude, the difference between the terms will always be a less fractional part of the greater term than it is of the less term; but as this difference continually decreases it will become the same fractional part of the less term that it was before of the greater term. For instance: let the lesser of the proposed terms be fixed, but let the greater be variable;

D

let

let the less term be always 99, but the greater term at a certain point of time be 100; their difference at that particular time

is 1, which is a $\frac{1}{99}$ th part of the less term,

but is only an $\frac{1}{100}$ th part of the greater

term. Now as the difference between the terms is supposed to continually decrease (so as to become equal to or less than any proposed fractional part of the greater term) let the difference sink to 0,99 (so as to be-

come an $\frac{1}{101}$ th part of the greater term)

and it will now be just an $\frac{1}{100}$ th part of the

less term; that is, the difference will now be the same fractional part of the less term that it was before of the greater term. The terms which before were 100 and 99, will now be 99,99 and 99, and their difference which before was 1, will now be 0,99, or

an $\frac{1}{100}$ th part of 99: thus then if the dif-

ference be at any time an $\frac{1}{100}$ th part of the greater term, it will (afterwards) also be an

$\frac{1}{100}$ th part of the less term; if the difference

ence become a $\frac{1}{1000}$ th part of the greater term, it will also become a $\frac{1}{1000}$ th part of the less term; and if the difference between the two quantities vanishes in respect of one of them, it does also vanish in respect of the other.

22. We may hence enlarge our definition par. 7, and in laying down the third condition say, the difference must vanish in respect *either* of the fixed, or of the varying quantity, since one of these implies the other.

23. What was laid down in par. 21, is true, though neither of the terms be fixed, but both of them increase or decrease without limit perpetually. In the instance before considered par. 18, in which the terms were $x + d$ and x , and where x continually increasing, the terms themselves do both of them continually increase. In this instance if the difference vanishes in respect of one of the terms it will also vanish in respect of the other. Let x be to d at any assigned point of time as n is to 1; and d at that instant of time will be the $\frac{1}{n+1}$ th part of

the greater term, and the $\frac{1}{n}$ th part of the less term; now though the cotemporary value of these fractions can never be equal, yet in succession (as x and consequently n increases for ever without limit) the value of the latter fraction will become whatever

the former has been. If $\frac{1}{n+1}$ becomes less

than any assigned fraction so will $\frac{1}{n}$ likewise; and thus if the difference d vanishes in respect of one of the terms $x + d$ (because x increases without limit) so will d vanish also in respect of the other term x . The same would be true if the terms were x and $x - d$ all things being as before.

24. Again; if x by decreasing vanishes in respect of some fixed quantity a , then will x multiplied by any given number n or nx vanish in respect of a ; or which is the same if x vanish in respect of a , so likewise will the quantity nx , bearing to x the assigned (or as it is called the finite) ratio of n to 1. For though at any particular point of time, x is a smaller fractional part of a than nx , yet x can be no assigned fractional part of a whatever, but by further

ther diminishing x the quantity nx may become the same fractional part of a that x was before. If then x may become equal to, or less than any assigned fractional part of a , so likewise may nx ; that is, if x vanishes in respect of a so likewise does nx .

25. And for a like reason, if x vanish in respect of any quantity a , it will likewise vanish in respect of that quantity multiplied or divided by any number; or it will vanish in respect of a quantity bearing to a any assigned ratio, as that of m to 1. Thus if x vanish in respect of a , it will also vanish in respect of $3a$, $2a$, or $\frac{1}{2}a$, or $\frac{1}{3}a$. If any line vanish in respect to the diameter of a circle, it will likewise vanish in respect of the radius. For whatever part of the diameter x may be at any assigned point of time, let x further decrease till it be half what it was at that assigned time, and it will now be the same part of the radius that it was before of the diameter. By these rules we may frequently find when one quantity vanishes in respect of another proposed quantity.

26. We have seen before that the ratio which two quantities bear to each other may have a limit although the terms themselves

selves may increase or decrease perpetually without limit. If the terms approximate to each other, if the less never pass the greater, and lastly, if their difference vanish in respect of either term,* then the limit of their varying ratio is that of equality; and this whether the terms themselves are such as by increasing do both of them become greater than any assigned quantity, or which is more common, such as by decreasing become less than any assigned quantity, or as it is usually called, *infinitely little*. Because the idea of the terms of a ratio is less abstract, than that of the ratio itself, it is more usual to say that the terms themselves in this case are ultimately equal, though strictly it is the ratio of the terms only that comes to a limit; for the terms themselves are here supposed to increase or decrease without limit, or as we commonly say in infinitum.

27. That the difference between two quantities may thus vanish in respect of those quantities; even though the quantities themselves do vanish in respect of some assigned (or finite) quantity we shall show by the following examples.

LEMMA

* See par. 20, 21. 23.

L E M M A

28. The angle of reflection is equal to the angle of incidence.*

P R O B L E M

29. Let a ray of light parallel to the axis EC (fig. 3) of a spherical concave speculum fall on a given point A : To determine the intersection of the reflected ray with the axis of the speculum

Let AC be the speculum, E the center, EC the axis, QA the incident ray parallel to the axis EC and meeting the speculum in A , let AT be the reflected ray intersecting the axis in T ; draw the radius EA to the point of incidence and because the angles AET and QAE †, also QAE and EAT ‡ are equal, the triangle EAT is isosceles. Draw DT perpendicular to EA and it will bisect the radius EA in D . Lastly draw AP the sine of the arch AC whose versed sine is PC . Then we have $EP : EA :: ED : ET$; call $EA = r$, $PC = v$, and $ED = \frac{1}{2}r$, therefore $r - v : r :: \frac{1}{2}r : \frac{\frac{1}{2}r r}{r - v} = ET$, and v vanishing $ET = \frac{1}{2}r$. Therefore

* Smith's Optics, Art. 8. † 29 El. 1. ‡ Lemma par. 28.

fore bisect the radius EC in F , and if the incident ray QA be supposed always parallel to the axis EC and continually to approach to it, the intersection T of the reflected ray with the axis will continually approach to the point F as its limit.

30. This is sometimes proved thus, $EP : EA :: ED : ET$, but EP is to EA , ultimately in a ratio of equality * therefore ultimately $ET = ED$ or half the radius. This will in some sort appear from the nature of the figure itself, for while the incident ray approaches to the axis, the base EA of the isosceles triangle EAT continues the same, but its height TD continually diminishes; therefore the sides ET and TA will at last coincide with the base EA and because they are equal, ET will at last be equal to half the base or half the radius.

31. But as this way of reasoning may not seem very strict, to show the case clearly we will prove that the fixed quantity $EF = \frac{1}{2}r$ is the limit of the varying quantity $ET = \frac{\frac{1}{2}r \cdot r}{r - v}$.

The quantity ET always exceeds the limit EF , their difference FT being

* See par. 39.

$$= \frac{\frac{1}{2} r r}{r-v} - \frac{1}{2} r = \frac{\frac{1}{2} r v}{r-v}. \quad \text{Now first this}$$

difference continually decreases as (the arch AC and consequently) v decreases: *secondly*; it never becomes negative: *thirdly*; it vanishes in respect of the varying quantity

$$ET. \quad \text{For } ET : FT :: \frac{\frac{1}{2} r r}{r-v} : \frac{\frac{1}{2} r v}{r-v} :: r : v,$$

therefore as v vanishes in respect of r , so does FT in respect of ET the varying quantity; and therefore (as was before shown in par. 21) FT vanishes in respect of the fixed quantity EF .

32. We see then, that although the point T or intersection of the reflected ray with the axis, continually recedes from the speculum while the incident ray by moving parallel to itself approaches to the axis, yet this point cannot recede in infinitum, not even though the incident ray be supposed to approach infinitely near the axis. The point F is the *limit* to which the point of intersection may *approach* nearer than by any assigned distance, *beyond* which it cannot *pass*, and *at* which it can never *arrive* 'till (as we may say) the very intersection itself has lost its being, by the coincidence both of the incident and reflected ray with the axis.

33. This point F is called the *focus* of parallel rays, and the interval FT the *aberration* of the reflected rays from their *geometrical focus*. Now although FT vanishes in respect of EF or half the radius, yet it does not vanish in respect of PC the versed sine. On the contrary it approximates to half the versed sine as its limit, or to speak more strictly, it is the ratio of FT to $\frac{1}{2}PC$ that approximates to the ratio of equality. For the aberration FT being equal to $\frac{\frac{1}{2}r v}{r-v}$, and $\frac{1}{2}PC$ being equal to $\frac{1}{2}v$, their

difference is $\frac{\frac{1}{2}v v}{r-v}$ which never becomes

negative; moreover $\frac{\frac{1}{2}r v}{r-v} : \frac{\frac{1}{2}v v}{r-v} :: r : v$.

Therefore as v vanishes in respect of r , so does the difference between the aberration and half the versed sine in respect of the aberration and consequently in respect of half the versed sine too. Nor is it strange that the aberration which vanishes in respect of the fixed quantity $\frac{1}{2}r$, should yet come to a limit with the varying quantity $\frac{1}{2}v$, seeing this last quantity does likewise vanish in respect of $\frac{1}{2}r$.

34. If we were to represent this in the

com-

common concise way of speaking, we should say that the aberration $\frac{\frac{1}{2}r v}{r-v}$ when v vanishes becomes $\frac{\frac{1}{2}r v}{r}$ or $\frac{1}{2}v$, seeming to strike v out of the denominator of the fraction as being nothing, and yet retaining it in the numerator as being something. Indeed if v in the numerator be considered as nothing, the whole expression vanishes, and the aberration is nothing, or rather there is no aberration; and thus the whole design of finding out this aberration falls to the ground.

It is from conciseness of expression in such cases as this, that mathematicians have been accused of unfair proceedings; of making the same quantity in the same expression *to vanish or not to vanish* as suits the conclusion intended to be drawn from it: of considering quantities as something in the beginning of a demonstration, rejecting such of them as they please at the end of it, and yet retaining the conclusion drawn from the original supposition.*

35. Another proof of the concise kind is as follows. $EA:EP::ET:ED$, whence
by

* See Analyst, Section XIV and XV.

by division of proportion $EA : (EA - EP) : PC :: ET : ET - ED = ET - EF = FT$; alternately $EA : ET :: PC : FT$, but EA is to ET ultimately as 2 is to 1,* therefore ultimately $2 : 1 :: PC : FT$ therefore $FT = \frac{1}{2} PC$.

36. But to proceed. The intire aberration is as we have seen $\frac{\frac{1}{2} r v}{r - v}$, from this sub-

tract $\frac{1}{2} v$ and the difference $\frac{\frac{1}{2} v v}{r - v}$ may be

called the *error of the geometrical aberration*.

Now this quantity though it vanish in respect of the whole aberration yet comes to

a limit with respect to the quantity $\frac{v v}{2 r}$ or

a third proportional to the diameter and versed sine, or to use a more strict way of

speaking, this error $\frac{\frac{1}{2} v v}{r - v}$ is to $\frac{v v}{2 r}$ ultimately in a ratio of equality. For the

whole error of the aberration being $\frac{\frac{1}{2} v v}{r - v}$,

and the third proportional aforesaid being

$\frac{v v}{2 r} = \frac{\frac{1}{2} v v}{r}$, their difference is $\frac{\frac{1}{2} v^3}{r \times r - v}$,

and the whole error is to this difference as

r

* See par. 42.

r is to v , therefore this difference vanishes in respect of the whole error, and consequently in respect of $\frac{1}{2} \frac{v}{r}$. Thus then this third proportional to the diameter and versed sine may be called *the geometrical error of the geometrical aberration*. In like manner we may determine the error of this error and so on.*

37. To give an example of all this in numbers; let the radius of the speculum be 30 inches, the arch AC eleven degrees and twenty nine minutes, whose versed sine is 2 when the radius is 100, consequently when the radius is 30 inches this versed sine is 0,6 of an inch. We have therefore $r = 30$, $v = 0,6$, $\frac{1}{2} r r = 450$, $r - v = 29,4$, $\frac{\frac{1}{2} r r}{r - v} = \frac{450}{29,4}$ which (by common division) is $15,3061224 = ET$ the whole distance of the point of intersection from the center, and this exceeds $EF = \frac{1}{2} r = 15$, or the geometrical focus by 0,3061224. Again, this quantity which is the whole aberration exceeds the geometrical aberration or $\frac{1}{2} v = 0,3$ by 0,0061224. Again, the diameter being 60, and the versed sine 0,6, we have

60 :

* *Neque novit natura limitem.* Newton.

$60 : 0,6 :: 0,6 : 0,006 =$ the third proportional to the diameter and versed sine; but $0,0061224$ which is the whole error of the geometrical aberration, exceeds this third proportional by $0,000124$ and so on.

38. All this will be confirmed by dividing the numerator of the original fraction

$\left(\frac{\frac{1}{2} r r}{r-v}\right)$ by its denominator in the manner

of compound division in arithmetic, and so throwing the fraction into an infinite series. For the quotient will be $\frac{1}{2} r + \frac{1}{2} v + \frac{\frac{1}{2} v^2}{r} + \frac{\frac{1}{2} v^3}{r^2} + \frac{\frac{1}{2} v^4}{r^3} + \frac{\frac{1}{2} v^5}{r^4}, \&c.$ where the

law of the series is manifest, each term being to the subsequent one always in the ratio of r to v .

Let then v vanish in respect of r and all the terms of the series will vanish in respect of the first term; that is, the sum of all the terms will in this case approximate to $\frac{1}{2} r$ as the limit; therefore the geometrical focus is $\frac{1}{2} r$.

Subtract now $\frac{1}{2} r$ from the whole series expressing the distance ET , and we have

the whole aberration $FT = \frac{1}{2} v + \frac{\frac{1}{2} v^2}{r} + \frac{\frac{1}{2} v^3}{r^2} + \frac{\frac{1}{2} v^4}{r^3} + \frac{\frac{1}{2} v^5}{r^4}, \&c.$ Let now v again

be

be supposed to vanish in respect of r and all the terms in this series vanishing in respect of the first we have $\frac{1}{2}v$ for the geometrical aberration.

Once more; subtract $\frac{1}{2}v$ from the series expressing the whole aberration and we have $\frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$ for the whole error of the aberration. Let v again be supposed to vanish, and we have $\frac{\frac{1}{2}v^2}{r}$ for the limit of the error of the aberration, and so on.

Assuming $\frac{1}{2}r = 15$ and the ratio of r to v that of 100 to 2, as before we have

	$\frac{1}{2}r = 15,$	<i>Geometr. Focus.</i>
100 : 2 :: 15,0	: to $\frac{1}{2}v = 0,3$	<i>Geometr. Aberr.</i>
100 : 2 :: 0,3	: to $\frac{\frac{1}{2}v^2}{r} = 0,006$	<i>Geometr. Error.</i>
100 : 2 :: 0,006	: to $\frac{\frac{1}{2}v^3}{r^2} = 0,00012$	
100 : 2 :: 0,00012	: to $\frac{\frac{1}{2}v^4}{r^3} = 0,0000024$	

Sum of the terms = $ET = 15,3061224$
which is true to seven places of decimals as was before found by division.

PROPOSITION

39. In a circle whose center is C (fig. 4) radius CA , diameter Aa , let AB be the chord, FB the sine, AD the tangent of the arch AB . I say, that while the arch AB continually decreases without limit, *first* the sine continually approximates to the tangent. *Secondly* the sine never exceeds the tangent; *thirdly* their difference will vanish in respect of either sine or tangent.

First. $BF : DA :: CF : CA$, but while the arch AB decreases CF approximates to CA , therefore BF approximates to DA . *Secondly*. CF can never exceed CA , therefore BF can never exceed DA . *Thirdly*. The arch continually decreasing without limit vanishes (in our sense of the word par. 20) in respect of the diameter which is fixed; therefore the chord AB which is less than the arch, likewise vanishes in respect of the diameter Aa . But $Aa : AB :: AB : AF$, therefore if AB vanishes in respect of Aa , AF will vanish in respect of AB , and much more will AF vanish in respect of Aa : but if AF vanish in respect of Aa , it will also vanish in respect of $\frac{1}{2}Aa$ or CA (by par. 25). Now $CA : CF :: AD : FB$

FB , and by division of proportion, $CA : AF :: AD : AD - FB$. As therefore AF vanishes in respect of CA , so does $AD - FB$ (the difference of the sine and tangent) vanish in respect of AD the tangent, and consequently in respect of FB the sine, by par. 23.

We see then that while the arch continually decreases without limit, the sine approximates to the tangent; *secondly*, the sine never exceeds the tangent; *thirdly*, although the sine and tangent both vanish in respect of the radius, yet their difference vanishes in respect of these quantities themselves. Therefore the ratio of equality is the limit of the varying ratio which the sine and tangent have to each other, while they both decrease perpetually without limit; or it is their *ultimate ratio*; or we may say that they are *ultimately equal*.

40. The sine is less than the chord; for in the right angled triangle AFB , the side BF is less than the hypotenuse AB . The chord is less than the arch; this is self-evident, the chord being a straight line, and the arch a curve, both terminated by the same points A and B . The arch is less than the tangent. For from the point D draw

F
another

another tangent to the circle in H ; and the lines ABH , ADH will be terminated by the same points A and H , and will have their concavities turned the same way: therefore the included arch ABH will be less than the sum of the two equal tangents DA and DH ; * consequently half that arch or AB will be less than half the sum of the tangents or AD .

41. *Cor. 1.* Hence the fine chord arch and tangent are all ultimately in a ratio of equality. This may appear because the chord and arch are included between the fine and tangent, but perhaps more plainly thus. Of these four quantities, *viz.* the fine chord arch and tangent, the fine is the least, and the tangent the greatest, by the last paragraph; therefore the difference between the fine and tangent is greater than the difference between any other two of those four quantities. If therefore the greatest of all those differences, *viz.* that between the fine and tangent vanishes in respect of the least of all those quantities namely the fine; much more will the difference between any other two of those four quantities vanish in respect of the quantities themselves.

42. *Cor.*

* *Archimedes De sphaera & cylindro. Defin. 2 & axiom. 2* or *Mac Laurin's Fluxions, art. 180 and 183.*

42. *Cor. 2.* Join Ba and in the right angled triangle BFa , the hypotenuse Ba is greater than the side Fa ; therefore $Aa - Ba$ is less than $Aa - Fa$ or AF ; but by par. 39 while the arch AB decreases perpetually without limit, AF vanishes in respect of Aa ; much more then does $Aa - Ba$, the difference between Aa and Ba vanish in respect of Aa , therefore the ultimate ratio of Aa to Ba is that of equality; and the ultimate ratio of $\frac{1}{2}Aa$ or BC to Ba , is that of 1 to 2.

S C H O L I U M

43. The reader will find a different proof of this celebrated proposition in *Saunderson's Fluxions*, page 248, and another in *Morgan's notes on Robault*, published by Dr. S. Clarke, Part II. Chap. 28, and re-published at Cambridge 1770, see page 28. The best proof of all others is undoubtedly Newton's own proof. *Princip. Lib. I. Lemma VII.* That demonstration being very concise, I shall give it here, and enlarge upon the several steps.

Let ACB (fig. 13) be an arch of a circle whose center is R , chord AB , tangent AD , secant RD . Let the tangent AD be pro-

duced to a distant point d , and through d draw dr parallel to the secant DR , cutting the line AR (produced indefinitely) in r ; then if with the center r and radius Ar an arch $Ac b$ be described meeting dr in b , *first* it will be similar to the arch ACB , (though not similarly situated) * that is, the arches $Ac b$ and ACB will have each the same proportion to the circumferences of their respective circles, because the angles $Ar d$ and ARD are equal by construction. † *Secondly*, the chord AB produced, will pass through b , and ABb will be the chord of the arch $Ac b$: for the arches $Ac b$ and ACB being similar, their chords AB and Ab will make equal angles with their common tangent ADd and therefore coincide. *Lastly*, the lines AB , AD and arch ACB are always proportional to the lines Ab , Ad and arch $Ac b$.

For the line AB } $\frac{AB}{AD}$ { line Ab } as the radius AR , is to
the line AD } $\frac{AD}{Ad}$ { line Ad } the radius Ar , or as AD
the arch ACB } $\frac{ACB}{Ac b}$ { arch $Ac b$ } is to Ad .

therefore alternately $AB : AD :: Ab : Ad$,
and likewise $AB : ACB :: Ab : Ac b$: that
is, the lines AB , AD and arch ACB are
to each other, as the lines Ab , Ad and arch
 $Ac b$ are to each other. This premised,

imagine

* Mac Laurin's Fluxions, art. 122.

† 29 El. 1.

imagine now the secant RBD to turn on the center R , while the point B continually approaches to the point A . At the same time imagine the line dr to turn round the fixed point d in the contrary direction, but with the same angular motion so as always to keep parallel to RD ; by this means the center r will recede from A more and more in infinitum, and the arch $Ac b$ will be a portion of a circle continually growing larger and larger; or as we may say, the arch $Ac b$ will continually unbend itself. In the mean time the chord $AB b$ turning round the point A , continually increases, and as the line dr approaches to parallelism with the line Ar , the increasing chord Ab approximates to the invariable tangent Ad and carries the intermediate arch $Ac b$ along with it. Now the point B approaching to the point A the angle BAD continually decreases, and there is no limit to its decrease by *prop. 16. El. 3*, therefore neither is there any limit to the decrease of the angle bAd nor to the approach of Ab to Ad , therefore the lines Ab , Ad and the intermediate arch $Ac b$ will ultimately coincide and be equal; whence the chord AB , tangent AD and arch ACB , which are to each other as Ab ,
 Ad

Ad and $Ac b$, will be ultimately to each other in a ratio of equality.

We may observe, that the lines Ab , Ad are always finite. Ad is of a fixed invariable length; Ab approximates to Ad , can never exceed Ad , nor indeed become equal to it 'till db is parallel to AR ; but the finite lines Ab and Ad can approach nearer to equality than by any assigned difference; that is, the difference between Ab and Ad vanishes in respect of those (finite) lines; whence this is inferred of the lines AB , AD always proportional to them.

Newton's artifice consists in showing us a finite part of an infinitely large circle which we *can* see; and arguing from it analogically to an infinitely small part of a finite circle which we *can not* see.

The foregoing explanation restraining the Lemma to circles, puts a sense upon it more confined than Newton intended, but it will be the easier understood by beginners.

To propose the Lemma in more general terms, we must first premise, that the curve ACB is supposed to have continual curvature at A , and though not a circle, yet it is supposed to have circular curvature at A ,

A ,* that so the angle between the chord and tangent (while the point B approaches to A) may become less than any (assigned) right lined angle.

In the demonstration the point d is supposed to be fixed, the line Ad being of a given length. *Secondly*, The line bd is always to be parallel to the secant BD . *Thirdly*, The two curves ACB and $Ac b$ are to have one common tangent at A . *Lastly*, The arch $Ac b$ is to be always similar to the arch ACB , that is, ACB and $Ac b$ are to be similar parts of similar figures, the linear magnitude of $Ac b$ being to that of ACB , as Ad is to AD ; moreover, A represents corresponding points of the two figures. † For instance; if ACB be part of a circle, then must $Ac b$ be also part of a circle: The radius of $Ac b$ must be to the radius of ACB as Ad is to AD , and then the part $Ac b$, cut off by the line db , and the part ACB will be similar arches of two circles. If ACB be a Parabola then must $Ac b$ also be a Parabola, whose parameter is to the parameter of ACB , as Ad is to AD . If ACB be an Ellipse

* *Cæterum in his omnibus supponimus curvaturam ad punctum A , nec infinitè parvam esse, nec infinitè magnam. Vide Scholium ad Lemma XI. Lib. I. Princip.*

† *Mac Laurin's Fluxions, art. 122.*

Ellipse then must $Ac b$ be an Ellipse. To make make these Ellipses similar figures, the proportion of the greater axis to the less, in each Ellipse, must be the same; and that the arch $Ac b$, cut off by the line db , may be similar to the arch ACB , the greater axis of the latter Ellipse must be to the greater axis of the former, as Ad is to AD ; moreover, if the point A is the extremity of the greater axis in one Ellipse, it must likewise be so in the other.

It follows from the similarity of the arches ACB and $Ac b$, *First*, that their subtenses AB and Ab make equal angles with their common tangent ADd , and therefore coincide. *Secondly*, that the lines AB , AD , and arch ACB are always to each other as the lines Ab , Ad , and arch $Ac b$ are to each other. This premised, while the point B approaches to A , the chord Ab , turning round the point A , approximates to the invariable tangent Ad , and carries the intermediate arch $Ac b$ along with it. As the angle DAB decreases without limit, or in other words ultimately vanishes, so the angle dAb ultimately vanishes, and the finite lines Ab , Ad , and arch $Ac b$, ultimately coinciding, are equal; therefore the evanescent

nescent lines AB , AD and arch ACB , (which are always to each other as Ab , Ad and $Ac b$) will have for their ultimate ratio, that of equality.

The angle ADB is always supposed to be finite, but it may be either constant or variable. The angles ADB and $Ad b$ may if you please be supposed always right angles, or any other given angle; the demonstration will still be the same.

L E M M A I.

44. If a body be acted upon by a force which is every where as the distance from the center A , (fig. 5) the time of its descent to that center will be the same from whatever place it falls. *Princip. Lib. I. Prop. XXXVIII. Cor. 2.*

L E M M A II.

45. If a body be acted upon by a force which is every where as the distance from the center A , (fig. 5) the time of descent through any space PA , is to the time in which it would descend through that same space if impelled by half the first force (at P) uniformly continued, as the circumference of a circle is to four diameters.

For the time of descent through the space PA is equal to the time in which the body revolving by the force at P , in a circle whose radius is PA , would describe a quadrant, *Princip. Lib. I. Prop. XXXVIII. Cor. I.* and in this time by the same force uniformly continued, the body would descend through a space which is a third proportional to the diameter of the circle and quadrant thus described, *Princip. Lib. I. Prop. IV. Cor. 9*; that is calling the diameter D and quadrant

Q , a space equal to $\frac{Q^2}{D}$; but the time of

descent through this space $\frac{Q^2}{D}$ by the uni-

form force at P , is to the time of descent through the space $PA = \frac{1}{2}D$ by half that uniform force, in a subduplicate ratio of those spaces directly, and a subduplicate ratio of those forces inversely; * that is directly as

$\sqrt{\frac{Q^2}{D}}$ is to $\sqrt{\frac{1}{2}D}$, and inversely as $\sqrt{1}$ is

to $\sqrt{\frac{1}{2}}$, or compounding these ratios, as Q is to D , or as $4Q$ is to $4D$, or as the circumference of a circle is to four diameters.

* Introduction to *Saunderson's fluxions*. Prop. VII. Cor. IV.
Can. 3.

PROPOSITION I.

46. The accelerating force of a body oscillating in a circle, is to its absolute weight, as the sine of its angular distance from the lowest point, is to the radius.

Let P (fig. 6) be the body oscillating in the arch of a circle PAp , whose lowest point is A , and center C . Let PF be the sine of the arch PA , and let PT be a tangent to the circle at P meeting the radius CA produced in T , then will the accelerating force of the body at P be the same as if it were placed on the tangent PT considered as an inclined plane, and will therefore be to its absolute weight, as the height of that plane is to its length, * or as FT is to PT , or because of the similar triangles $PF T$ and $CF P$, as PF is to CP , or as the sine of the arch PA (the measure of the angular distance PCA) is to radius.

47. *Cor. 1*, In the same circle the accelerating force is every where as the sine of the arch PA , but because the sine and arch approximate to the ratio of equality while the arch decreases, therefore the ratio of the

* Introduction to *Saunderson's* fluxions, Prop. VII, or *Keil's* Physics, Lect. XV. Theorem 35.

the sines to each other will approximate to, and be ultimately equal to the ratio of their corresponding arches to each other. Wherefore ultimately, the accelerating force is as the arch PA , the distance of the body from the lowest point A measured along the circle in which it oscillates.

48. *Cor. 2.* Hence the times of descent down unequal arches PA, eA , approximate to equality while those arches decrease in infinitum, and (as we may say) ultimately those times will be equal. For when the accelerating force is as the distances PA, eA , the times of descent will be equal from whatever place the body begins to descend, by par. 44.

PROPOSITION II.

49. All things as before, draw PA the chord of the arch PeA (fig. 7); I say that the accelerating force of the body upon the arch at P , is to the accelerating force of the body placed any where* on the chord PA , ultimately as 2 to 1.

Draw the diameter aCA , join aP and CP , let AT be a tangent to the circle at A
and

* Introduction to *Saunderson's* fluxions, Prop. VII. Cor. I.
or *Keil's* Physics, Lect. XV. Theorem 35.

and PT a tangent to the circle at P , let these tangents meet in T ; then considering the chord PA and tangent PT as inclined planes whose heights are equal, the accelerating force on the plane PT (equal to that on the arch at P) will be to the accelerating force on the plane PA , as PA is to PT ,* or because of the similar triangles PAT and PaC , † as Pa is to PC , that is ultimately as 2 is to 1. ‡

50. *Cor.* 1. The time of descent through the arch PeA is to the time of descent through the chord PA , ultimately, as the circumference is to four diameters. For ultimately the length of the arch and chord are equal, and ultimately the accelerating force of the body on the arch is every where as its distance from the lowest point A , by par. 47; but the accelerating force of the body on the chord is uniform and equal to half the first force on the arch at P by this proposition, therefore by par. 45, the time of descent through the arch, is to the time of descent through the chord, as the circumference is to four diameters.

51. *Cor.* 2. The time of one oscillation is

* Introduction to *Saunderson's* Fluxions, Prop. VII. Cor. I. or *Keil's* Physics, Lect. XV. Theorem 35. † 32 El. 3, ‡ Par. 42,

is to the time of descent down half the length of the pendulum *ultimately*, as the circumference of a circle to its diameter. For the time of descent through the chord PA being always equal to the time of descent down the diameter aA ,* we have *ultimately*, the time of descent through the arch, to the time of descent down the diameter, as the circumference is to four diameters by the last corollary; whence the descent and ascent along the arch, or the time of one oscillation, is to the time of descent down the diameter, as the circumference to two diameters; but the descent down the diameter, is to the descent down $\frac{1}{4}$ of the diameter, or half the length of the pendulum, as 2 to 1; therefore (compounding these ratios) the time of one oscillation, is to the time of descent down half the length of the pendulum, as the circumference of a circle to its diameter.

SCHOLIUM

52. If the fines increased in the same ratio with the arches or distances PeA , all oscillations would be precisely equal; but as the fines increase in a less ratio than their arch-

* Keil's Physics, Lect, XV. Theorem 39.

arches, the accelerating force when the arch is enlarged, is not sufficiently increased to make the times of descent and consequently the oscillations equal; therefore the oscillations through larger arches take up a longer time. Contrariwise, if the arch be lessened, the time of descent is lessened, and therefore the time of descent and ascent, or the time of one oscillation is lessened. But though the arch may decrease in infinitum without any limit, yet the time of an oscillation through that arch will not decrease proportionably nor in infinitum, but has a limit. Thus if the length of the pendulum be 39,2 inches, the time of one oscillation through any arch will be somewhat more than one second. Now though that arch may be taken less than any assigned arch, yet the time of one oscillation will not become less than any assigned time. For if the pendulum oscillates at all, it must take up a determinate finite time, namely one second, how small soever the arch may be, through which it oscillates. Thus then although strictly speaking, there is no such thing as the time of a *least oscillation*, (there being no *least arch* and therefore no *least oscillation*;) yet there is strictly speaking a *least time* of the

the oscillation of every pendulum; and the proportion of this *least time* to the time of descent down half the length of that pendulum has been before pointed out.

53. The time here sought is that which is the limit of the varying time of an oscillation through an arch supposed to decrease continually without limit; therefore in comparing those quantities on which the time of one oscillation depends, it is the limit of their varying ratio, or rather the limit of the varying ratio of the length of those lines to which such quantities are analogous, that must be attended to. *

In the first corollary to the first proposition par. 47, when we had found the accelerating force to be always as the sines, we thence concluded it to be ultimately as the arches, because of the ultimate equality of the sine and arch. In the first corollary to the second proposition par. 50, we compared the time of descent along the arch, with the time of descent along the chord, and as far as those times depend on the length of the line described, (and not on the accelerating force) we considered them as ultimately equal,

* It may be observed, that in this and similar cases, the very nature of the problem shows that it is the limit of the varying quantity, or the limit of the varying ratio that we are to seek; as will be further exemplified in par. 55.

equal, because of the ultimate equality of the length of the arch and chord. In both these cases then we admitted this proposition concerning the ultimate equality of the sine chord and arch ; and it was rightly applied because the ratio of the quantities to be compared, (the forces in the former case, par. 47, and the times in the latter, par. 50) depended on the ratio of the *length* of these lines ; but we misunderstand the proposition, if we suppose it to imply that the arch and chord do ultimately *so coincide* that they may be considered as *one and the same line*. For on whatever grounds we suppose the arch thus to coincide with the chord, we might likewise suppose it to coincide with the sine PF , or with the tangent to the lowest point TA (fig. 8) for these are all *ultimately equal* ; but both sine and tangent in this case are parallel to the horizon and therefore there would be no accelerating force, no oscillation at all ; and thus the whole enquiry would fall to the ground. On the contrary ; the arch is a curve all whose parts have different inclinations to the horizon, but the chord is a right line which has every where the same inclination. The accelerating force on the arch continually

H

nually

nually increases with the distance from the lowest point, but the force on the chord is every where the same. Neither the law of the accelerating force, nor the quantity of it at the beginning of the descent in these two cases approximate by the decrease of the arch; we have therefore no reason to expect that the times of descent should approximate, notwithstanding the lengths of the lines descended through, do approximate and become ultimately equal. *Keil* and others after him say that the *arch and chord differ but little either in length or declivity, therefore neither do the times of descent.** It should be shown that the times are in some finite ratio of these quantities, otherwise small differences in these circumstances may occasion great differences in the times of descent. The assertion (rightly explained) is true with regard to the length of the chord and arch, but not with respect to their declivity, if we allow that any comparison can be made between the declivity of a curve and a right line, which yet seems to be absurd.

54. The ratio of the length of the arch
to

* *Keil's* Physics, Lect. 15. Theorem 41. *Monf. Parent*, I believe was the first that fell into this error.

to the sine, or of the arch to the chord which varies while the arch decreases, *has for its limit the ratio of equality*, the difference of their lengths vanishing in respect of the lengths themselves, all which is only expressed in other words, when we say that they are *ultimately equal*; but the arch is ever to be considered as a CURVE, and the chord as a RIGHT LINE, two ideas which ought not to be confounded. *

55. We said before that in problems of this

* Newton says — *figura rectilinea, quæ chordis evanescentium arcuum comprehenditur, coincidit ultimo cum figura curvilinea. Princip. Lib. I. Lemma III. Cor. 2.* — This is the express language of indivisibles, and must not be understood literally any more than *summa ultima* in the preceding corollary. Newton himself expounds *summa ultima* and *ratio ultima* by *limites summarum & rationum*, and rests the whole force of his demonstrations upon the doctrine of LIMITS. *Scholium ad Lem. XI. pag. 37. Ed. 3.*

Newton frequently expresses himself in such terms as were used by the writers on indivisibles in his time; and though at the end of that remarkable scholium he cautions his reader against being misled by the use of those terms, yet he seems sometimes to forget that caution himself, of which these corollaries are an instance. The very expression — *ultimæ quantitatum evanescentium rationes*, (or ultimate ratios) taken literally implies the ratios of quantities in an ultimate state, that is, of quantities infinitely small or infinitely great: and though Newton afterwards says that by ultimate ratios is meant the limit of the varying ratio of decreasing quantities, and not the ratio of ultimate quantities, yet he still speaks of varying ratios as reaching their limit — *quum quantitates diminuuntur in infinitum*, — when the terms become infinitely little.

It will be said that all this is only to cavil about words; — so it is, with those who interpret rightly this mode of speaking; but it is what will certainly mislead common readers, and almost as certainly prove a snare to the very ablest writer, if he is not continually upon his guard.

fort, their nature shows that it is the limit of some varying ratio that we are to seek, not the ratio itself. Thus if the areas of two curvilinear spaces are to be compared together, let their bases be each divided into the same number of parts and on those divisions let parallelograms be drawn so as to form right lined figures inscribed within the curvilinear ones, as is directed in the fourth lemma of the first book of the *Principia*. Now both the ratio of these parallelograms (each to each) and of the right lined figures composed of them all will vary, as the number of parts varies into which the base of each figure is divided, and the limit of the varying ratio (when it can be found) is that of the curvilinear figures themselves, as is proved in the lemma aforesaid, and a very neat example of finding that limit in a particular case is given by *Saunderson* in his fluxions, page 243. The nature of the problem (*viz.* to find the ratio of the curvilinear figures) shows plainly that it is not the ratio of the inscribed rectilinear figures that we are to seek, how great soever the number of the little parallelograms that compose them may be, but the limit of that (varying) ratio.

56. It

56. It may be worth while to illustrate this by another instance in the solution of the following

P R O B L E M

To find the proportion of the velocities of two flowing lines at a given instant, from their known cotemporary increments.

Let us for the present suppose the velocities of these lines to be continually accelerated; now if the time in which their cotemporary increments are generated be supposed perpetually to decrease, the ratio of those increments will continually vary; and the limit of their varying ratio is that of the velocities sought.

For every such increment may be distinguished into two parts; *first*, that which generated by the velocity of the flowing line at the given instant, continued uniformly for the whole time in which the increment is acquired; and *secondly*, that part of it which is generated in consequence of the subsequent increase of the first velocity. Now this additional velocity grows gradually from nothing, therefore by lessening the time in which the increment is produced, this additional velocity will continually

nually lessen, and will become less than any assigned fractional part of the first velocity which the line had at the given instant. Hence the latter part of the increment, generated in consequence of the additional velocity, will ultimately vanish in respect of the former part of the increment generated by the first velocity immutably continued, even though by thus diminishing the time the former part of the increment does itself vanish in respect of any assigned or finite line. In any such flowing line then, if the whole increment be compared with that part of it which is generated by the first velocity uniformly continued, *first* these two quantities continually approximate to each other as the time in which they are generated decreases; *secondly*, the one can never pass beyond the other, for the whole increment can never become less than a part of it: *lastly*, their difference, namely the part generated in consequence of the acceleration, vanishes in respect of the quantities themselves; therefore by par. 26, the limit of their varying ratio is that of equality.

In each line then the whole increment is to the former part of it (generated by the
velo-

velocity at the given instant uniformly continued) ultimately in a ratio of equality, and alternately the whole increment of one flowing line is to the whole cotemporary increment of the other flowing line, ultimately, as the former part of the increment of the first flowing line is to the former part of the increment of the other flowing line, that is, as the velocity of the first flowing line at the given instant, is to the velocity of the other flowing line.

57. Here the nature of the problem directs us to seek the *limit* of the varying ratio of the cotemporary increments of these lines, (varying as the time in which they are generated decreases perpetually without limit) not to seek the ratio of the increments themselves, how small soever we may suppose the time to be in which they are generated.

58. Having shown this in general, we may now proceed to enquire what the limit of that varying ratio is in any particular given instance. For example, let the former of these two lines be called x , and let the latter line be a third proportional to a fixed line (assumed at pleasure) called unity and to the former line x , then will its value
be

be xx ; that is, x and xx will represent any two cotemporary values of lines having this particular relation to each other. Let v be an increment of the line x , then will $x + v$ and $\overline{x + v}^2 = xx + 2xv + vv$ be two other cotemporary values of these flowing lines, whence their cotemporary increments will be v and $2xv + vv$ whose ratio is that of 1 to $2x + v$. Now supposing the time in which the increment v is generated to decrease continually without limit, the value of v will likewise decrease without limit, and will ultimately vanish in respect of the finite line $2x$, which is not affected by lessening the increment v , so that $2x + v$ will be to $2x$ ultimately in a ratio of equality. Hence the limit of the varying ratio of 1 to $2x + v$ while v decreases without limit, will be that of 1 to $2x$, which is therefore the proportion of the velocities sought. Therefore it will be, as the fixed line called unity is to twice the flowing line x , so is the velocity of the line x to the cotemporary velocity of the line xx .

59. The whole argument will be the same (*mutatis mutandis*) if we suppose the velocities of these lines to be continually retarded. The increment may in that case be

be distinguished into two parts, that which would be generated by the first velocity uniformly continued and that which is lost by the retardation. Now when the time is continually diminished, this latter part will ultimately vanish in respect of the former part as before. If we suppose the velocities of the flowing lines to be uniform, then their cotemporary increments are the measures of those velocities.

60. *Cor.* If the velocity of the line x be represented by $\dot{x}$, the cotemporary velocity of the line xx will be represented by $2x\dot{x}$. For 1 is to $2x$ as $\dot{x}$ is to $2x\dot{x}$.

ESSAY II.

On the POWER of the WEDGE.

1. **L**ET PEH (fig. 9) be a rectangular wedge, whose angular point is E , face PE , base EH , and back PH . Let this wedge slide freely on the plane LN . Let a body E be urged in the direction KE against the face of the wedge, and let that body be kept in the direction KE either by bearing against a plane parallel to KE , or by a string EB always drawing the body at right angles to KE ; or let the body be in any manner prevented from moving along PE and compelled to move in the line KE , by a force acting at right angles to KE .

2. Given the angle of the wedge PEH , and the angle KEP which the direction of the force upon E makes with the face of the wedge: To find the proportion between
the

the force applied to the body in the direction KE , and the force applied perpendicularly to the back of the wedge, when these two forces balance each other, so that the wedge moves neither forward nor backward. To find likewise the proportion of the forces aforefaid, to the pressure of the wedge against the plane LN and to the pressure of the body E against the plane parallel to KE , occasioned by the action of the other two forces when they balance each other.

Before we enter upon the solution of the problem, it may be proper to premise the following mechanical principles universally received.

3. *First*, When a body is acted upon by one force only, it cannot be sustained by another single force, unless the latter be both equal and *opposite* to the former.

4. *Secondly*, When two or more forces act at once upon a body, the combined effect of them all is to move it in one direction only; and they will have the same effect and be equivalent to a single force acting in one particular direction. *

5. *Thirdly*,

* The rule for finding the quantity of that force and the particular direction in which it must act, is in almost every book of mechanics. See the introduction to *Saunderson's fluxions*.

5. *Thirdly*, When an hard body acts *in any direction whatever* on the surface of another, and such surface is perfectly hard and smooth; whether the former body presses continually on the latter, or only strikes it, yet the effect of that pressure in one case, or of the impact in the other, and the consequent motion in the body *acted upon*, (supposing it at liberty,) is always in a DIRECTION PERPENDICULAR TO THAT SURFACE. *

6. These principles laid down it is manifest that when the body *E* is both urged in the direction *KE* and at the same time sustained in the line *KE* by a plane parallel to *KE*, the original force in the direction *KE* combined with the reaction of the sustaining plane (or with the force of the string *EB* drawing at right angles to *KE*) may be considered as a single force urging the body *E* against the face of the wedge: but the action

* When a plane perfectly hard and smooth is sustained against the pressure of an hard body, it is said to *react* against the body that presses it, and as the sustaining force must be in a direction opposite to that in which the plane would move if at liberty, the reaction of the plane is therefore perpendicular to its surface.

The force that sustains such a plane must be equal to the force of the body that presses it, that is, the reaction of the plane against the body, is equal to the action of the body upon the plane.

The principle here laid down is further explained in par. 21.

action of an hard body E , against the smooth and hard surface of the wedge is always perpendicular to its face PE , and a single force impressed on the wedge in the opposite direction, *and in that only*, will counteract the other two (combined) forces and keep the wedge in equilibrio. Now the directions of these three forces being known, their proportions when they balance each other are easily found by a common rule in mechanics. For through P draw PA parallel to KE , through E draw ED perpendicular to KE meeting PA in D , also draw EA perpendicular to PE meeting PA in A ; and ED , DA , EA the sides of the triangle EDA , lying in the direction of the forces aforesaid are to each other as those forces. *

7. EA

* It may perhaps throw some light upon this subject if we consider the whole effect of the original force in the direction KE , and the whole effect of the reaction of the sustaining plane, separately from each other.

Now when a force acts upon the body E in a direction KE oblique to the hard and smooth surface PE , such a force will produce in the body E a tendency to move along that surface PE , (down the face of the wedge) and likewise a pressure of the body E against the wedge, perpendicular to its face PE . The body E is here supposed to be kept from moving down the face of the wedge by another force, namely the reaction of the sustaining plane: This force is perpendicular to the former force KE , therefore likewise oblique to the surface PE . This force then, being in the oblique direction ED , partly urges the body E up the face of the wedge, and partly adds to the perpendicular pressure of the body E , against the wedge.

7. EA then represents the quantity and direction of the force impressed on the wedge by the body E , but the effort of the wedge to move in the direction EA is not in our case sustained by a single force equal and opposite, but by two others, one the reaction of the plane on which the wedge slides and the other the force on the back. The force EA being partly opposed to the plane LN , and partly to the back PH , will produce partly a pressure of the wedge upon the plane LN which is balanced by the reaction of that plane, and partly a tendency in the wedge to move backwards, which is balanced by the force applied to the back. Now the directions of these two forces as well as that of EA are known, whence their proportions when they balance each other will be thus found. From A
draw

wedge. From D let fall Dd perpendicular to EA ; and if DA as before represents the original force applied to the body E in the direction KE , then will Dd be the force urging the body down PE , and dA the force urging the body against the wedge. Again, if the sustaining plane prevents the body from moving down the face of the wedge, that part of the reaction of this plane which is in the direction EP must be equal and contrary to Dd , and therefore is represented by dD ; whence ED will represent the whole reacting force of that plane, and Ed will represent the other part of this force which urges the body perpendicularly against the face of the wedge. To Ed add dA , and their sum EA , will be the whole force with which the body E presses the face of the wedge, as before.

For the nature and laws of oblique forces, see par. 21.

draw AF perpendicular to EH ; and EA representing as before the quantity and direction of the force impressed on the wedge; FA will represent the pressure on the plane LN , and FE the force to be applied perpendicularly to the back of the wedge to keep it in equilibrio.

8. If then DA represents the force applied to the body E in the direction KE ; FE will be the force applied perpendicularly to the back of the wedge when these two forces balance each other: FA is the pressure of the wedge upon the plane EH , and DE the pressure of the body against the plane parallel to KE (or it is the tension of the string EB) occasioned by the action of the two former forces when they balance each other.

9. The right angled triangles AED , AEF , having one common hypotenuse AE , are similar to the right angled triangles EPD , EPH , having one common hypotenuse EP ; the sides of the former being respectively perpendicular to the sides of the latter. Therefore DA , FE , FA and DE , are respectively as ED , PH , EH and PD , or making EP radius, as the sine of $EPD = KEP$; the sine of PEH ; the cosine

cosine of PEH ; and the cosine of EPD , or KEP ; so that the force applied to the body in the direction KE , is to that on the back of the wedge which balances it; as the sine of KEP the angle of direction of the force upon E , to the sine of PEH the angle of the wedge: The pressure on the plane LN is represented by the cosine of the latter angle, and the pressure against the plane parallel to KE , by the cosine of the former angle.

10. *Cor. 1.* Let PA cut EH in I , and draw PK parallel to EH meeting EK in K , and while the wedge moves upon the plane NL in the direction NL or PK through the space $PK = IE$, the body E will be raised in its proper direction through $EK = PI$: therefore their respective contemporary velocities estimated in the directions in which the forces on each are respectively applied, are as EI to PI , or because of the similar triangles EDI , PIH , as ED to PH , or reciprocally as those forces.

11. *Cor. 2.* When KE is perpendicular to PE , the point D coincides with P , and $ED = EP$, and PD is nothing; that is, the pressure against the sustaining plane (as we called it) is nothing. The other three forces

forces are as EP , PH , and HE , the sides of the wedge to which they are respectively at right angles. This is the case commonly considered by writers on mechanics.

12. *Cor. 3.* When KE is parallel to PH , the points D and H coincide, and $ED = EH$, and the force on the body in the direction KE , is to that on the back of the wedge which balances it, as EH is to PH . PD is likewise equal to PH , that is, the pressure against the sustaining plane is equal to the force applied to the back of the wedge.

13. *Cor. 4.* If the wedge be in the form of an isosceles triangle ABC (fig. 10) whose sides are AC and BC , back AB , perpendicular height CH , and be applied to separate two bodies E and e in directions EF , and ef parallel to the back but opposite to each other, which bodies are kept in their respective directions by bearing against a plane, or by a string in the manner before laid down; then when the wedge is in equilibrium the perpendicular force on the back, will be to each of the (equal) forces FE or fe on the sides, as the whole back AB , is to CH the perpendicular height of the wedge.

K

This

This will appear by considering such a wedge as is composed of two rectangular ones, ACH and BCH . In this case the pressure of the two wedges ACH and BCH against one another (in the directions AH and BH) being equal and opposite will destroy each others effect. Now FE the force on the side AC , will be balanced by a perpendicular force on the back, which is to FE as AH is to CH by par. 12; and another perpendicular force on the back equal to the former will balance fe the force on the other side BC . Therefore the whole perpendicular force on the back, which balances both forces on the sides, is to one only of those forces as 2 AH or AB is to CH .

SCHOLIUM I.

14. The case last described is exactly that of the machines generally made use of to show experimentally the power of the wedge. In that described by 's *Gravesande* in the first and second editions of his experimental philosophy; by *Hawksbee* and *Whiston* in their course of lectures; by *Desaguliers* in his course of experimental philosophy, Lect. 3. par. 58, in all these the rollers to be separated by the wedge, rest
upon

upon the edge of a brass frame which determines the direction in which they must move.

These rollers are drawn together by weights tied to strings which passing over a pulley run along the edge of the frame, and therefore the weights act upon the rollers in the direction of that edge. The brass frame being set level, the wedge is put between the rollers, its back being parallel to the frame and to the horizon. The wedge is then loaden with weights till it begins to separate the rollers, which will be when the weight of the wedge itself and its load, is to the weight with which each roller *separately* is drawn along the frame, as the back of the wedge is to its perpendicular height.

15. In the machine described by 's *Gravesande*, in his third edition, N^r 279, and by Mr. *Ferguson* in his lectures, page 68. Ed. 1760, the rollers instead of resting upon the horizontal edge of a brass frame, are suspended by long lines which not only support the weight of these rollers, but also determine the direction in which they must move when separated, namely at right angles to these lines; that is, in a direction

parallel to the horizon as in the former machine.

16. It seems not to be observed that when the wedge is put between the rollers, the reaction of the brass plane on which they rest in the former machine, and the tension of the suspending lines in the latter, is increased by the whole weight of the wedge and its load. This would appear very plain in the latter machine if the lines by which the rollers are suspended, passed over pulleys instead of being fixed to hooks in the ceiling: For the rollers might then be balanced so as to hang suspended in equilibrio before the wedge is put between them. In such a case when the wedge is put in, that equilibrium will be destroyed, and to restore it, the rollers must be drawn upwards by the addition of a weight equal to the wedge and its load.

17. It therefore appears plainly that in making this experiment the rollers are not only drawn horizontally, but both of them upwards; that the rollers are each of them acted upon, not by one oblique force only, as these authors suppose, but by two; that the action of the wedge against the rollers (when they thus balance each other) does
of

of necessity create an additional tension of the fixed lines in 's *Gravesande's* latter experiment, and an additional pressure upon the brass plane in his former experiment; and lastly, that the quantity of this additional action of the rollers upon the brass plane and its consequent reaction, is exactly what we determined it to be in par. 12.

18. The two forces which thus act upon each roller are both of them oblique to the side of the wedge but lie on opposite sides of the perpendicular, their directions being at right angles to each other. Now these two forces will of necessity ASSUME such a proportion that the force compounded of them both shall be perpendicular to the side of the wedge otherwise there could be no equilibrium. For the force compounded of them both is sustained by another single force, namely the reaction of the side of the wedge, whose direction is at right angles to this surface, and upon this principle the proportion of these three forces was determined in par. 6. Thus then the direction in which the rollers act against the wedge, is always perpendicular to its sides however it may seem otherwise. A machine may be made in which an equilibrium shall
be

be produced when two forces act upon each roller in directions oblique to the sides of the wedge; because the rollers may in this case be urged perpendicularly against those sides; but no machine can be made in which *all* the parts shall be at rest, when ONE force only acts on each roller in a direction oblique to the side of the wedge. See par. 21. and 28.

19. Whilst we are upon this subject it may not be amiss to take notice of one or two other circumstances in these experiments which have occasioned much needless disputation. In all the machines but that described by 's *Gravesande* in his third edition, the strings which draw the rollers together run over fixed pulleys; that is, fixed to the wall of the room or to some firm body not connected with the rollers. Each weight then shows the force acting upon the roller it draws along, and the sum of the two weights is the sum of the two forces acting upon the rollers. In the machine described in 's *Gravesande's* third edition, a pulley is fastened to one of the moveable rollers, and a string being tied to the other roller is put over that pulley, at the end of which is hung a single weight by which
both

both rollers are drawn together horizontally. In this last case each roller is acted upon by the whole of that single weight, and the sum of the two forces acting upon the two rollers is twice that single weight. Now what shall we understand by the force with which the rollers *cobere*? or with which they are *drawn together*? or what shall we understand by the *resistance* to be overcome? shall we understand by it, the force with which each roller singly is drawn along the brass frame? or shall we understand by it the sum of the two forces by which the rollers are drawn along the brass frame in opposite directions, and which is double the former force? We have shown in par. 13, that in the former sense of the words the force on the back of the wedge, (or as it is often called, the *power*) is to the *resistance*, as the whole back of the wedge is to its perpendicular height; and therefore taking *resistance* in the latter sense, the *power* will be to the *resistance* as the whole back to twice the perpendicular height or as *half the back to the perpendicular height*. All the difficulty here is about *words*; by what names we should, or rather by what names we *do* usually call things, and if the language

guage held on these occasions be not settled by an uniform use of the terms among writers, no wonder that confusion should follow. Settle but the meaning of the terms and all disputing vanishes; but dealers in this sort of philosophy, not suspecting that they are talking to one another in a different language, *make experiments* to prove, as they say, the truth of their propositions. The experiments on both sides always succeed, and to our surprise, the controversy is no nearer to an end. — Why? — Because these philosophers are making experiments to find — *the meaning of a word.* *

SCHOLIUM II.

20. Most of the writers on mechanics, in demonstrating the power of the wedge, make use of the rule mentioned by Newton, “*that in mechanical engines, those powers balance each other, which are reciprocally as the velocities of the parts of the engines to which they*

* The dispute about the FORCE OF BODIES IN MOTION, in which all the philosophers in Europe were engaged for half a century together, was undoubtedly of this sort. See *An essay on quantity* by Mr. Reid, in the Philosophical Transactions, N^o 489, Vol. XLV, for the year 1748, or Martyn's Abridgement, Vol. X. Part 1. page 22. This little essay, hitherto unnoticed, well deserves the attention of every one who would understand either mathematical or philosophical subjects clearly.

they are applied, estimated according to the respective directions in which the powers act." *

Newton takes notice of the analogy of this rule to the rules concerning the collision of bodies, but gives no proof of it; nor does *Mac Laurin*, who mentions it from Newton, attempt to demonstrate it. † Newton himself demonstrates the fundamental proposition about the lever, by the resolution of forces; a doctrine ultimately referred by him to the first and second laws of motion, which laws he lays down as first principles or general axioms from which all particular kinds of motion are to be mathematically derived. ‡ The best writers following Newton's

* *Principia. Cor. VI. ad Leg. mot. pag. 26. Ed. 3.*

† *Mac Laurin's account, &c. Book. II. Chap. 3. Sect. 2.*

‡ Or as we call it *solved*. To solve a phenomenon is only to derive it from, or show that it is included in the general rules of motion. Thus we reduce that vast variety of motions we see in the world into a few general classes, and show that the universe thus regulated by a few simple laws is the effect of supreme intelligence. To solve a phenomenon then, is not to assign its cause. When we say that gravity is the *cause* of the moon's motion, we mean only that its motion is of the same sort with that of other bodies near the earth; that the moon's motion is conformable in its circumstances to certain laws of motion which obtain universally; as to the *cause*, for aught we know, an angel may carry it about.

Newton sufficiently warns his readers, (if they will take any warning) against this common conceit that philosophy has to do with causes. See *Principia. Def. VIII. at the end. Page 6. Ed. 3.* Want of attention to this has introduced a great deal of absurd metaphysics into natural philosophy, of which the reader may see enough in a paper by *Dr. S. Clarke*, on the force of bodies in motion. *Philos. Trans. N^o 401.*

ton's example, demonstrate the other mechanical powers from the same principles,* but in treating of the wedge, always restrain their proof to a particular case where the forces are applied at right angles to the sides of the wedge. It may be said, perhaps, that three forces only (one to each side) cannot be otherwise applied so as to make an equilibrium. Let us see however what determination the theory gives in one of the simplest cases of oblique forces, namely, when two equal forces act on the sides of an isosceles wedge in directions parallel to the back but opposite to each other, and are sustained by a third force acting perpendicularly on the back of the wedge.†

21. Before we enter upon the solution
of

* *Mac Laurin's* account, &c. *Morgan's* notes on *Robault*, republished at *Cambridge* 1770.

The futility of the common proof, drawn from what is called the equality of *momentums*, is sufficiently shown by *Dr. Hamilton* in his ingenious essay on the principles of mechanics.

† *Dr. Hamilton* in his essay, page 165, proposes to consider the case of oblique forces, but his demonstration, if conclusive, respects a very different one. In this demonstration the protruding force EP (fig. 11) is considered as equivalent to two others GP and PE ; that is, the perpendicular reaction of the side of the wedge against the body E will be sustained by the joint action of the two oblique forces GP and EG upon the body: True — but then we suppose the body E to be sustained not by one, but by two forces, both of them (separately) oblique to the side of the wedge, but conjointly perpendicular; both necessary to make an equilibrium, therefore neither can properly be said to *be lost*.

of this problem, it may be proper to premise an account of the nature and laws of these oblique forces.

Strictly speaking, there is no such thing in the abstract as an oblique force, or a force producing a motion in a *single body* not in the direction of that force. The motion produced in every body is supposed to be in the direction of the force that produces it. Indeed the direction of the motion produced is what marks out the direction of the force producing it, and the quantity of this motion is the measure of that force. But one force only may produce a motion in **TWO** bodies (acting against each other) neither of which motions *taken separately* shall be in the direction of the original force producing them. Thus one body may be urged against (or may strike) the surface of another in a direction oblique to that surface, and then (supposing them hard) the motion produced in each of these ~~two~~ bodies (considered separately) will neither be in the direction of the original force, nor will its quantity be so great as what the original force alone would produce. To determine the effect of such an oblique action of one body against another, the original force is to

be

L 2

be resolved into two, one parallel and one perpendicular to the surface of the body acted upon. The separate effect of these two forces will thus be manifest, and by the laws of motion, the effect of two such forces is the same as that of the original force alone. Now the quantity and direction of the motion produced in *the body acting against the plane* is such as would be produced by the parallel force alone. The perpendicular force indeed urges the body against the plane (whose reaction sustains that force) but creates no motion in this body along the plane.* Again; the quantity and direction of the motion impressed on *the body whose surface is acted upon*, is such as would be produced by the perpendicular force alone, the parallel force creating no pressure of the *acting body* against that plane. Having thus laid down the rule for finding the direction of the forces impressed on these bodies separately and their proportion to the original force, we proceed to the solution of the problem.

22. Let

* If both these bodies be not hard, and their surface smooth, this parallel motion of the body along the plane will be prevented by the indenting, or by their roughness, that is, by the action of these bodies against each other. The original force in such a case cannot generate a separate motion in each body; both will then have one common motion just as if one body had pressed or struck the surface of the other perpendicularly.

22. Let ABC (fig. 10) be the isosceles wedge, whose angular point is C , sides AC and BC , back AB , perpendicular height HC : Let FE represent the quantity and direction of the force applied to one of the sides; resolve this into two other forces, FD and DE , the former parallel, the latter perpendicular to the side AC , and the original oblique force FE will have just the same effect upon the wedge as a lesser perpendicular force DE , the former being to the latter as AC is to HC . But this last perpendicular force on the side AC is to that on the back which balances it, as AC is to AH , by par. 11; whence compounding these ratios, the oblique force against one side of the wedge is to the perpendicular force on the back which balances it, as AC^2 is to $AH \times HC$.

The oblique force fe on the other side of the wedge being equal to FE , will require another perpendicular force on the back to balance it, equal to the former perpendicular force; whence the whole force on both sides of the wedge, is to the whole force on the back, as AC^2 is to $AH \times HC$, or as the square of the side of the wedge to the rectangle under half the back and the perpendicular height.

23. There

23. There is no paralogism in this demonstration, and the conclusion would obtain in fact, if a machine could be constructed in which all the circumstances should exactly agree to those here supposed, that is, to the *data* of the problem. If for instance, two rollers could be made to press the sides of the wedge in directions parallel to the back, but opposite to each other, and at the same time be permitted to roll down the sides of the wedge with part of their original force; — but nothing like this is the case of the machines before mentioned, though intended to show experimentally the power of the wedge in the circumstances laid down in this problem. Therefore the experiments made with those machines, though they undoubtedly succeed just as related, yet do by no means prove the proposition they are brought to prove, nor do they invalidate the reasoning here advanced. On the other hand, the problem, as it is here proposed, perhaps will not suit any case in which the wedge can be *practically* introduced, but must ever remain a matter of *useless speculation*.

APPENDIX.

What follows is a description of two machines intended to show the power of the wedge in the case before mentioned. How far their circumstances answer to the *data* of the problem in par. 20, is left to the reader's consideration.

24. Let a roller F , (fig. 11) be drawn by a string DF running over a pulley D , so fixed, that when the roller bears against the side BC of the isosceles wedge ABC , the string DF shall be parallel to BC . At the same time let this roller be drawn in the direction FE by another string FE running parallel to AC the back of the wedge. Let another roller E be applied in like manner to AB the other side of the wedge. Lastly, let the force with which the rollers act against the sides, be balanced by another force P acting perpendicularly to AC the back of the wedge: then will the sum of the two forces on the sides be to the force on the back, as AB^2 is to $AP \times PB$.

The rollers are here supposed to have no weight of their own, but to owe their whole force to weights that draw them.

25. The roller F is here acted upon by
two

two forces, one of which being parallel to BC , opposes the motion of the roller along the side CB , but adds nothing to its pressure against the side of the wedge. The roller is also drawn in the direction FE , oblique to the side BC , but it can move only in the direction FP , perpendicular to the string FD , and therefore perpendicular to the side of the wedge.

26. Another machine better adapted to practice is described by Mr. *Ferguson* in his lectures, page 67. Ed. 1760. Plate VI. Fig. 6. and is here delineated in Fig. 12. In this machine the wedge ABC is not isosceles but rectangular. The string FG (which must be always kept parallel to BC) may be either fixed to an hook, or made to run over a pulley H , in which case the cylinder F must be poised by a weight W . The equilibrium will be when the weight of the cylinder F is to the weight K , that draws the wedge along the horizontal plane (in the direction AI , perpendicular to the back AB) as BC^2 is to $AB \times AC$; as will be shown presently in par. 29.

27. The cylinder F is here acted upon by two forces, the weight W and the force of gravity; the former only opposes the de-

descent of the cylinder down BC the face of the wedge, and creates no pressure of the cylinder against that face. The force of gravity acts in a direction parallel to the back BA , and therefore oblique to the face BC ; yet while the wedge is drawn along the horizontal plane, the cylinder F constantly rises in a line which is (relatively to fixed objects) perpendicular to the face BC , and can move in no other direction.

28. If the cylinder was suffered to descend freely down the face of the wedge BC , and the wedge was kept from running backwards (during the time of descent) by the weight K ; the case would then exactly answer the *data* of the problem, in par. 20, excepting that there would be here one oblique force only; the other two being perpendicular to the sides of the wedge on which they act.

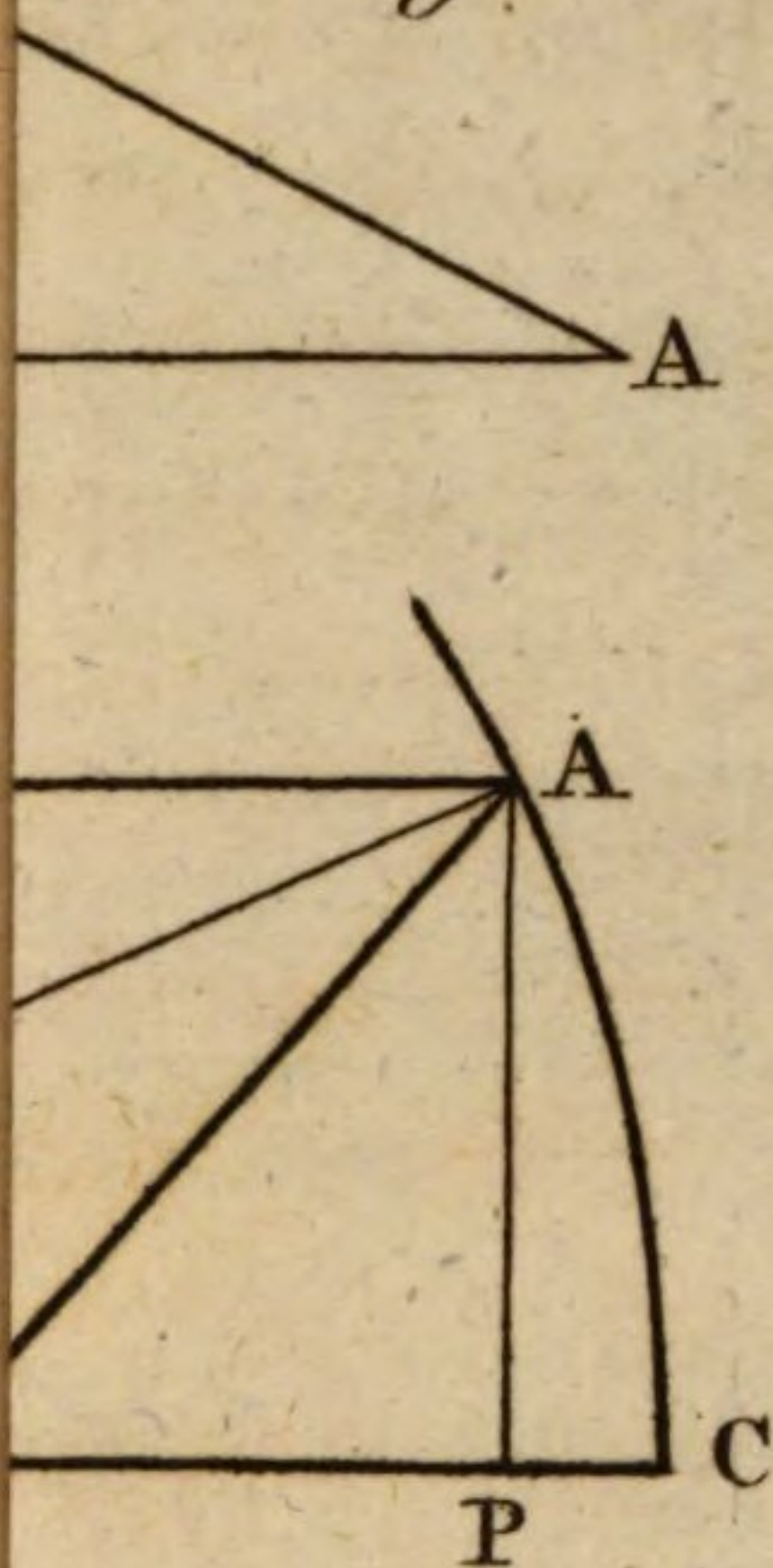
29. From A let fall AD perpendicular to BC , and from D let fall DE perpendicular to AC . Let now BA represent the weight of the cylinder, then the force (of gravity) BA may be resolved into the forces BD and DA : Again; the force DA may be resolved into the forces DE and EA . Now BA , BD , DE and EA , are to each other

other as BC^2 , $AB \times BC$, AC^2 , and $AB \times AC$ respectively: therefore if BC^2 represents the weight of the cylinder, then $AB \times BC$ will represent the force with which it descends down the face of the wedge; AC^2 will be the pressure of the wedge upon the horizontal plane, and $AB \times AC$ the force impelling the wedge (backwards) along the horizontal plane.—This last force will be greatest when AB and AC are equal, in which case it will be half the weight of the cylinder.

30. Let the string be fixed at H , and let the wedge be drawn under the cylinder; then when the wedge has been drawn its whole length AC on the horizontal plane, the cylinder F will be raised along the face of the wedge from C to D , whose perpendicular height, above the horizontal plane AC , is DE ; therefore the cotemporary velocities of the weights F and K estimated in the directions in which they act, are reciprocally as AC to DE , that is, in a ratio compounded of AC to AD , and AD to DE ; or as BC^2 to $AB \times AC$, that is, reciprocally as those weights when they balance each other.

T H E E N D.

Fig. 2.



d

Fig. 3.

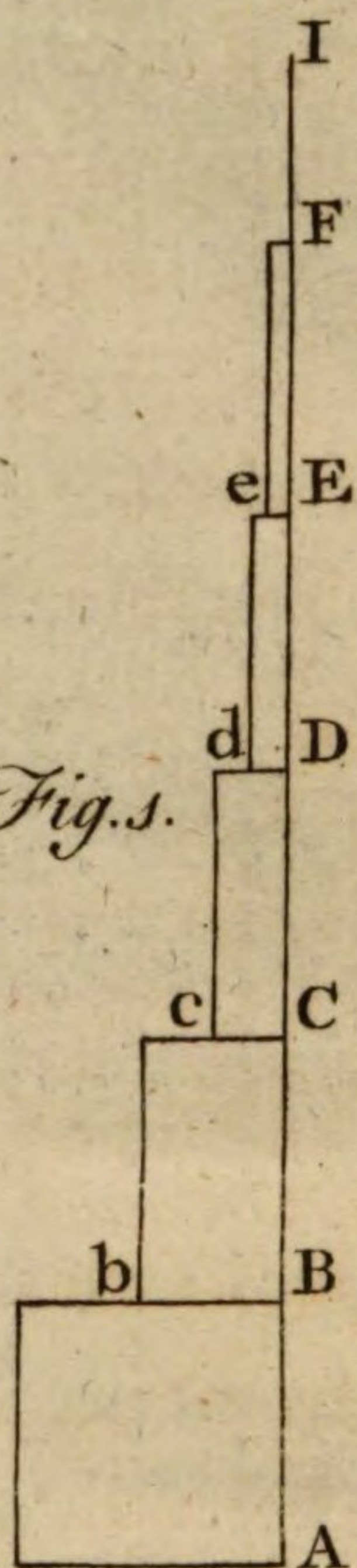
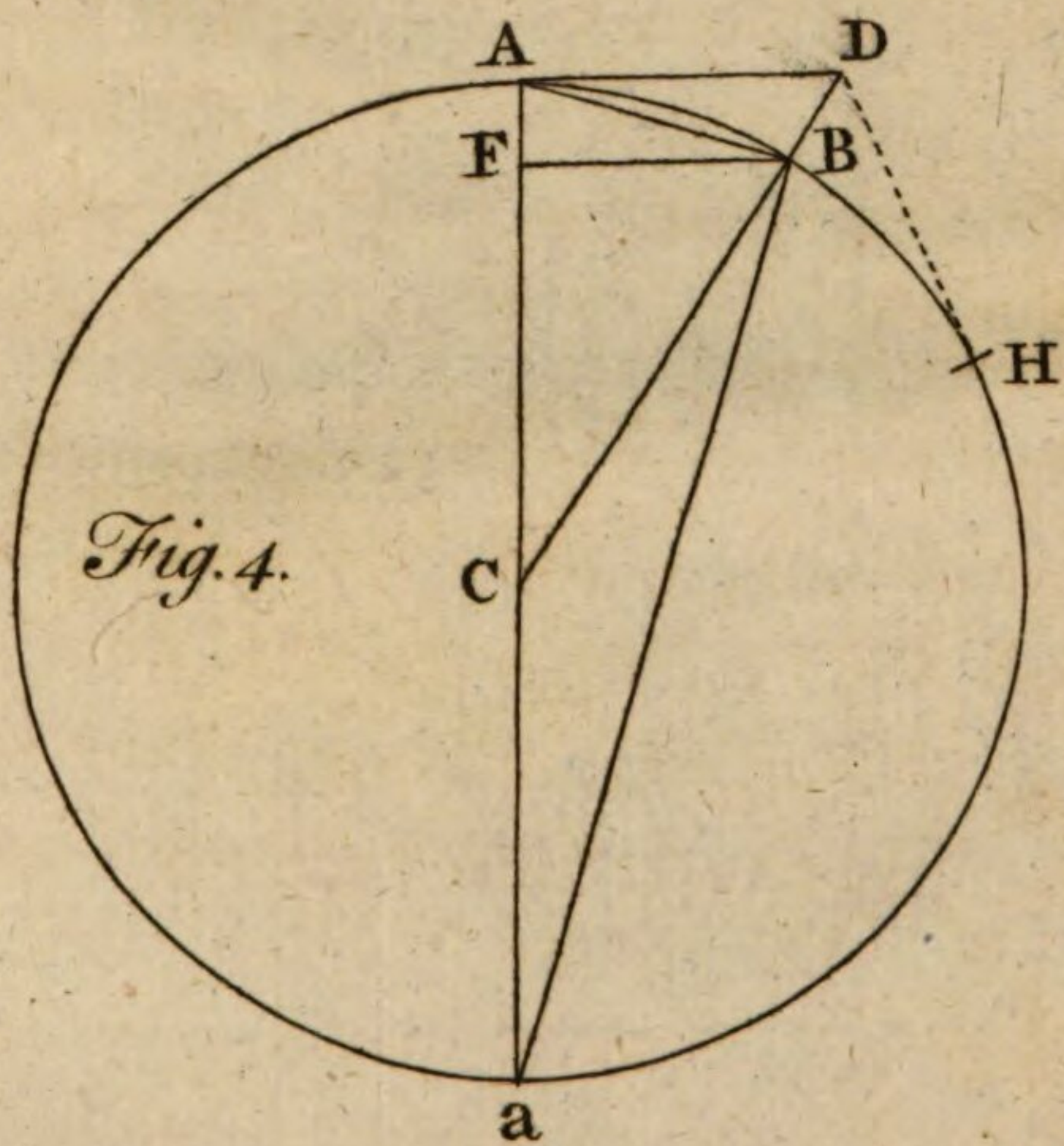
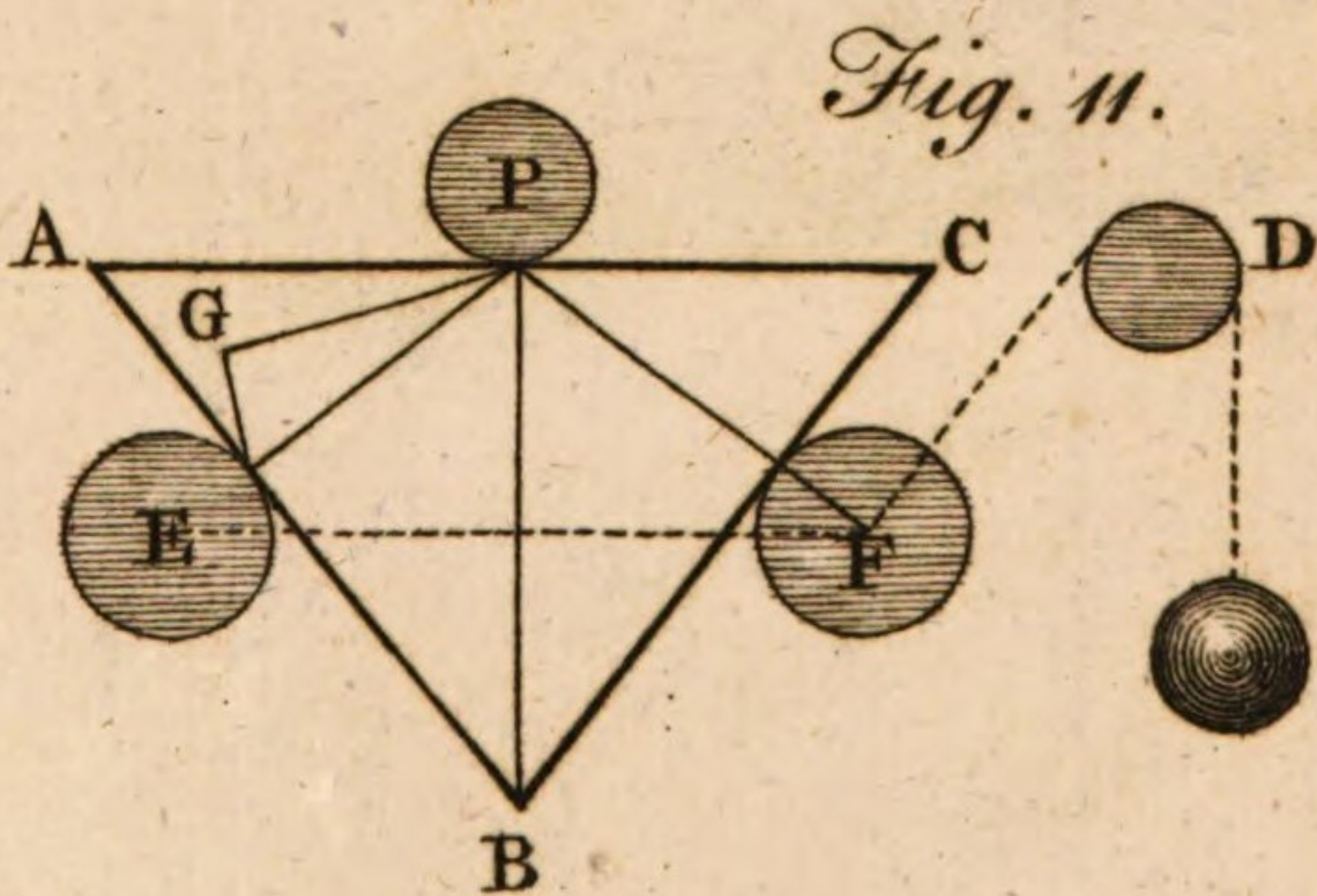
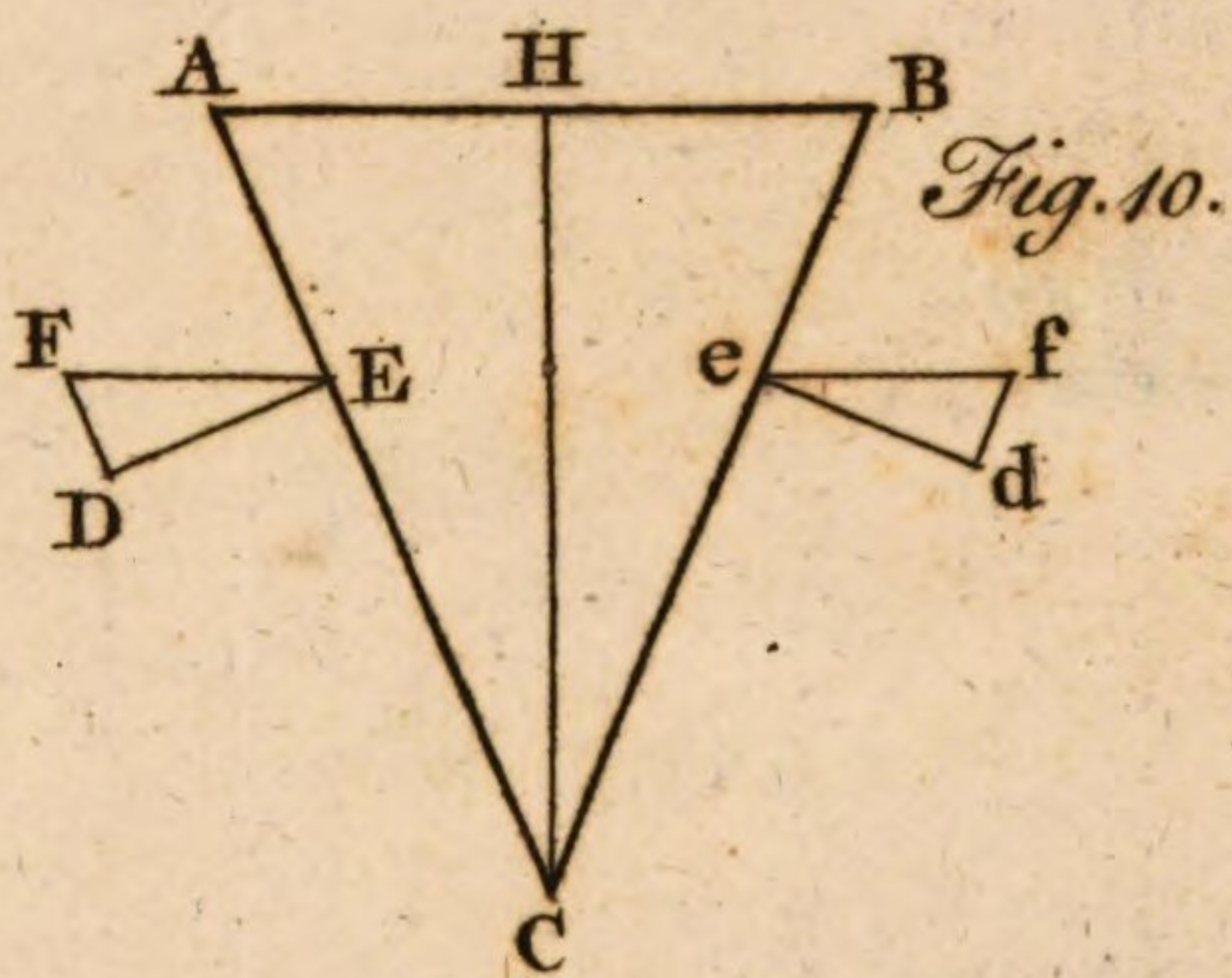
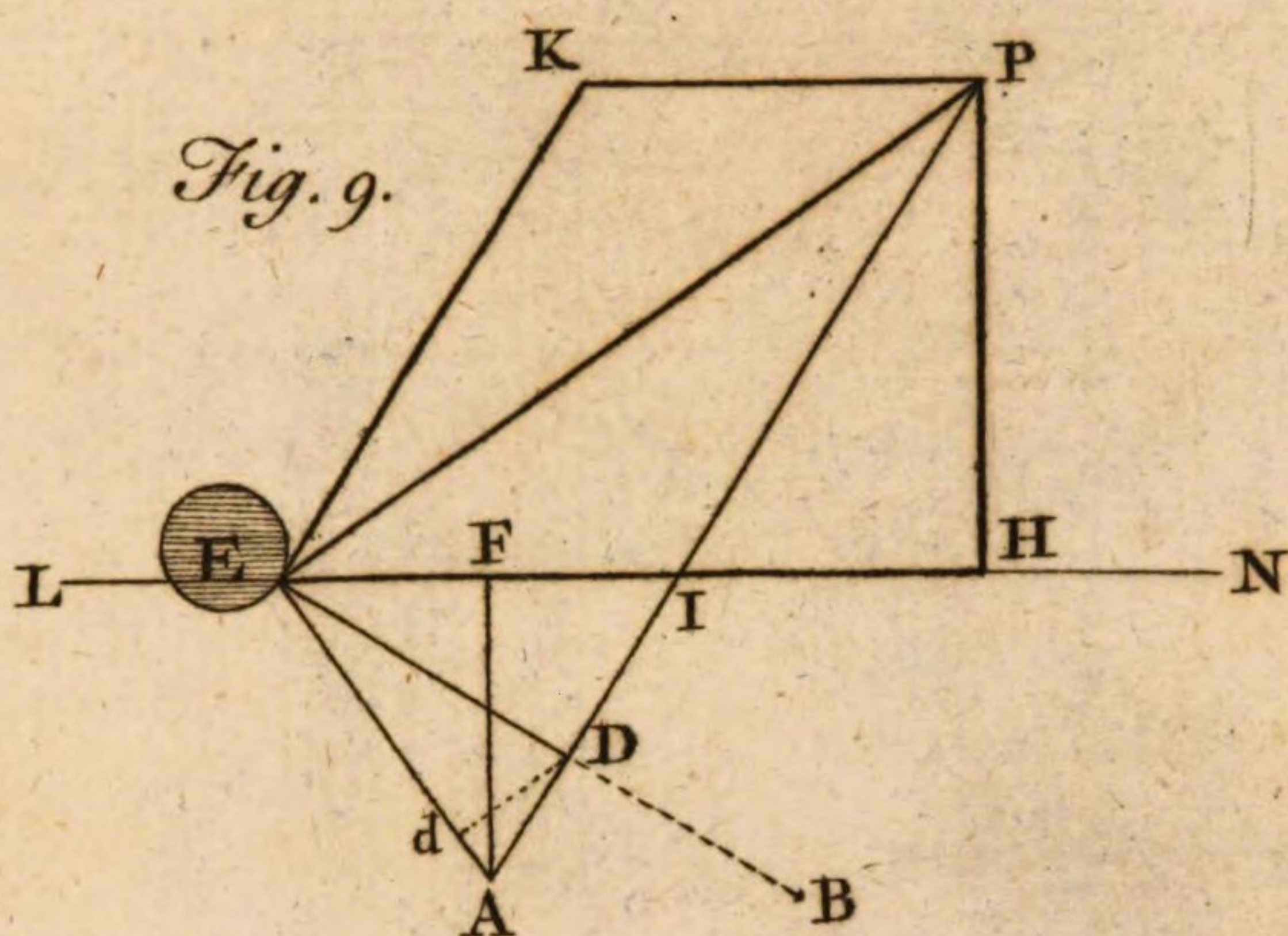


Fig. 4.



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J. Mynde, sc.



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Two mathematical Essays

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